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Unitary matrix model of a chiral $[\mathrm{SU}(N)]^K$ gauge theory*

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Abstract: We build a matrix model of a chiral $[\mathrm{SU}(N)]^K$ gauge theory (SQCD_5 deconstructed down to 4D) using random *unitary* matrices to model chiral bifundamental fields $(\mathbf{N}, \bar{\mathbf{N}})$ (without $(\bar{\mathbf{N}}, \mathbf{N})$). We verify the duality by matching the loop equation of the matrix model to the anomaly equations of the gauge theory. Then we evaluate the matrix model's free energy and use it to derive the effective superpotential for the gaugino condensates.

Keywords: Duality in Gauge Field Theories, M(atrix) Theories.

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Contents

1. Introduction

Three years ago, Robbert Dijkgraaf and Cumrun Vafa discovered a peculiar duality between the $\mathcal{N}=1$ supersymmetric gauge theories in 4D and the bosonic matrix models without any spacetime at all. At first [1, 2], they showed that the effective superpotential for the gaugino condensates and the abelian gauge couplings follow from planar diagrams in the gauge theory (see also [3, 4]), and then they argued [5] that the very same perturbative series also gives the free energy of a bosonic random matrix model whose action is similar to the tree-level superpotential of the gauge theory. Shortly afterwards, Cachazo, Douglas, Seiberg, and Witten [6 − 8] pointed out that the planar diagrams describe the on-shell chiral ring of the gauge theory. They used a different technique to study the on-shell ring, namely the generalized Konishi anomaly equations, and those equations turned out to be exactly similar to the loop equations of the random matrix model. This confirmed the Dijkgraaf-Vafa gauge-matrix duality and made it more precise: the matrix model is dual to a subring of the gauge theory's chiral ring comprising the gaugino condensates and the mesons; other operators of the gauge theory are invisible to the matrix model. In particular, the sphere-level free energy of the matrix model is dual to the effective prepotential of the gaugino condensates.

Gauge-matrix duality works for all kinds of theories, including quiver theories with multiple gauge groups and bifundamental matter [2]. Here is a brief summary of Dijkgraaf-Vafa rules for building the matrix model of a particular gauge theory:

1. A $\mathrm{U}(n)$ or $\mathrm{SU}(n)$ gauge symmetry of the field theory becomes a $\mathrm{U}(\hat{N})$ global symmetry of the matrix model, and we take the $\hat{N}\to\infty$ limit. If multiple gauge symmetries $\mathrm{SU}(n_i)$ are involved, we take all the $\hat{N}_i\to\infty$ at the same rate: $\hat{N}_i = t\times n_i$ for the same $t\to\infty$. Similar rules apply to $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$ gauge group factors.

2. Chiral superfields becomes bosonic variables of the matrix model in similar multiplets of the symmetry group. Thus quark and antiquark superfields become complex vectors of length $\hat{N}$, adjoint superfields become $\hat{N} \times \hat{N}$ matrices, and the bi-fundamental multiplets of an $\mathrm{SU}(n) \times \mathrm{SU}(m)$ symmetry become $\hat{N} \times \hat{M}$ matrices.

3. A *complex* chiral superfield becomes a *real* bosonic variable of the matrix model (or rather t or t^2 real variables. Thus a complex adjoint superfield becomes a *hermitian* $\hat{N} \times \hat{N}$ matrix. Likewise, a complex vector A of the matrix model and its conjugate $B = A^\dagger$ together represent both the quark and the antiquark superfields of the gauge theory.

Rule **3** works well for non-chiral gauge theories with real matter multiples (or conjugate pairs like $\square + \overline{\square}$) but not for chiral theories. For example, consider an $\mathrm{SU}(n) \times \mathrm{SU}(m)$ theory with a *chiral* bifundamental field $(\mathbf{n}, \overline{\mathbf{m}})$ without the conjugate $(\overline{\mathbf{n}}, \mathbf{m})$ multiplets.[1] By rules **1** and **2** this field becomes an $\hat{N} \times \hat{M}$ matrix transforming as a bifundamental of the $\mathrm{U}(\hat{N}) \times \mathrm{U}(\hat{M})$ symmetry, and since this representation is complex, the matrix must be complex too. But according to rule **3** such complex $\hat{N} \times \hat{M}$ matrix corresponds to the non-chiral $(\mathbf{n}, \overline{\mathbf{m}}) + (\overline{\mathbf{n}}, \mathbf{m})$ multiplet of fields rather than the chiral $(\mathbf{n}, \overline{\mathbf{m}})$ without the $(\overline{\mathbf{n}}, \mathbf{m})$.

Lazaroiu et al. [9, 10] found a way out of this problem: instead of real bosonic variables integrated over $\mathbb{R}$, one may use holomorphic variables, *i.e.* complex variables integrated over some contours in the complex plane rather than the whole plane $\mathbb{C}$. More generally, one integrates $\hat{N} \times \hat{M}$ matrix elements of a complex matrix over some variety $\Gamma \subset \mathbb{C}^{\hat{N}\hat{M}}$ of *real* dimension $\hat{N}\hat{M}$. The variety Γ must be consistent with the symmetries of the matrix model according to rules **1** and **2**. For non-chiral models one may identify Γ with the real "axis" $\mathbb{R}^{\hat{N}\hat{M}}$ of $\mathbb{C}^{\hat{N}\hat{M}}$ and recover rule **3**, but chiral models require non-linear integration varieties. Lazaroiu et al. found such non-linear Γ for the model of a single-$\mathrm{SU}(n)$ gauge theory with chiral $\begin{smallmatrix}\square\\\square\end{smallmatrix} + (n-4)\,\overline{\square}$ matter spectrum [10], but in this article we interested in a different theory.

Specifically, we are interested in the chiral bifundamental fields, hence we are looking for varieties of $\hat{N} \times \hat{M}$ complex matrices which are invariant under the $\mathrm{U}(\hat{N})_L \times \mathrm{U}(\hat{M})_R$ symmetry action. To be precise, the variety Γ should satisfy

$$\forall\, V_L \in \mathrm{U}(\hat{N})\,,\ \forall\, V_R \in \mathrm{U}(\hat{M}) : \quad V_L \Gamma V_R^\dagger \cong \Gamma \tag{1.1}$$

where '$\cong$' means equivalence as an integration variety: same topology with respect to the singularities of the integrand, and similar asymptotics when one or more matrix elements approach the complex infinity. For square matrices $\hat{N} \times \hat{N}$ there is a simple solution, namely the unitary group space $\Gamma = \mathrm{U}(\hat{N})$ — which is actually invariant under the symmetry (1.1) ($V_L \Gamma V_R^\dagger = \Gamma$) and has the right real dimension $= \hat{N}^2$. The problem is much more difficult for the rectangular matrices with $\hat{M} \neq \hat{N}$, and so we leave them for future research. In this article, we work with the square matrices integrated over $\Gamma = \mathrm{U}(\hat{N})$.

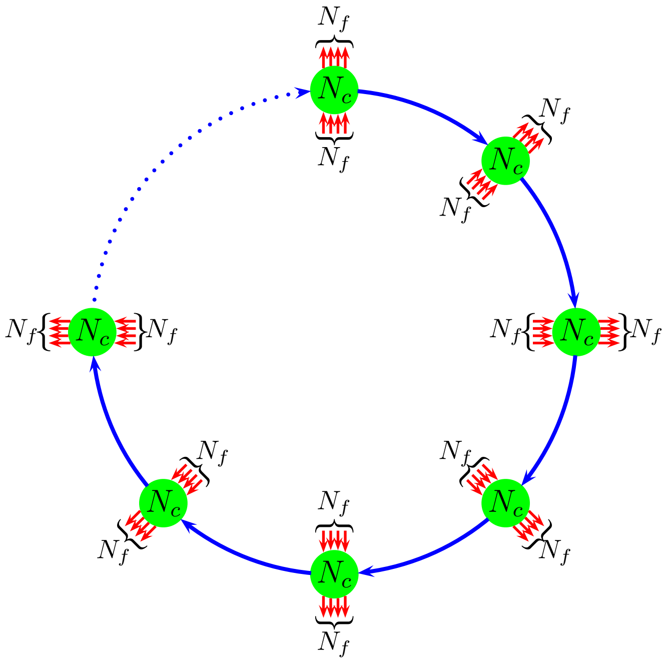
[1] The $(\mathbf{n}, \overline{\mathbf{m}})$ bifundamental must be accompanied by other chiral multiplets with opposite anomaly, but the present argument does not depend on those multiplets.

On the gauge theory side, a unitary matrix $\mathrm{U} \in \mathrm{U}(\hat{N})$ is dual to an $(\mathbf{n}, \bar{\mathbf{n}})$ bifundamental field Ω with non-zero eigenvalues. That is, the vacuum states of the field theory which have matrix-model duals must have $\langle \det \Omega \rangle \neq 0$; there may also be vacua with $\langle \det \Omega \rangle = 0$, but the matrix model might not work for such vacua. Indeed, under the gauge-matrix duality, the allowed values of the fields correspond to matrix variables belonging to the integration variety Γ or any of its allowed deformations $\Gamma' \cong \Gamma$. For example, a bosonic variable z integrated over a unit circle with measure $\oint \frac{dz}{z}$ can also be integrated over any other loop surrounding the $z = 0$ origin. However, no such loop can go through the $z = 0$ point itself because of the measure singularity, and consequently the field φ dual to z can take any complex values except $\varphi = 0$. Likewise, the unitary matrix integral can be deformed to integral over a variety $\Gamma' \cong \mathrm{U}(\hat{N})$ using an analytic extension

$$\int_{\Gamma'} d\omega[\mathrm{U}] = \int_{\Gamma'} \prod_{\substack{(i,j) \\ \text{pairs}}} (\mathrm{U}^{-1}\, dU)_{i,j} \tag{1.2}$$

of the Haar measure. However, this extension becomes singular for $\det \mathrm{U} = 0$, which limits the Γ' integration varieties to the invertible matrices only. Hence, on the gauge theory side of the duality we should have $\det \Omega \neq 0$, otherwise the duality might break down.

The bifundamental fields with $\det \Omega \neq 0$ are common in dimensional deconstruction. Accordingly, in this article we build a unitary matrix model of an $[\mathrm{SU}(N_c)]^K$ 4D supersymmetric gauge theory which deconstructs the 5D SQCD [11]; the chiral ring of this theory was studied in great detail in [12]. The matter fields of the theory are shown on the following quiver diagram:



(1.3)

In particular, the blue lines here denote chiral bifundamentals, which we shall model via unitary matrices U_ℓ, $\ell = 1, 2, \ldots, K$. The red lines denote the quark and the antiquark fields; we shall model them in the usual way as complex $\hat{N} \times \hat{F}$ matrices A_ℓ and $B_\ell = A_\ell^\dagger$.

The article is organized as follows: in the next section 2 we build the matrix model of the gauge theory (1.3) and establish the gauge-matrix correspondence. In particular, we derive the the loop equations of the matrix model and see that they agree with the Konishi anomaly equations of the gauge theory. In section 3 we evaluate the matrix integral and derive the free energy of the matrix model in terms of contour integrals on the spectral curve. We begin by reducing the problem to an integral over a single unitary matrix, and then we adapt the technology of Dijkgraaf and Vafa [5, 2, 1] and Cachazo et al. [6] to the unitary case. In section 4 we derive the effective superpotential for the off-shell gaugino condensates of the dual gauge theory. Finally, in section 5 we discuss open questions related to the present research.

2. The unitary matrix model and its loop equations

2.1 The $[\mathrm{SU}(N_c)]^K$ gauge theory and its matrix model

Dimensional deconstruction of 5D SQCD leads to an $\mathcal{N} = 1$ $[\mathrm{SU}(N_c)]^K$ gauge theory in 4D as described in [13, 11]. The chiral ring of this $[\mathrm{SU}(N_c)]^K$ theory was analyzed in much detail in [12], and in this article we build and study its matrix model. We begin with the basic structure of the 4D field theory as shown in the quiver diagram (1.3): the green circles denote simple factors of the net gauge group

$$G_{4\mathrm{D}} = \prod_{\ell=1}^{K} [\mathrm{SU}(N_c)]_{\ell} \tag{2.1}$$

while the red and blue arrows denote the chiral superfields:

$$\begin{aligned} \Big\}\ \text{quarks } Q_{\ell,f} &= (\square_{\ell}), \\ \Big\}\ \text{antiquarks } \widetilde{Q}^{f}_{\ell} &= (\overline{\square}_{\ell}), \\ \longrightarrow\ \text{link fields } \Omega_{\ell} &= (\square_{\ell+1}, \overline{\square}_{\ell}), \end{aligned} \tag{2.2}$$

where $f = 1, 2, \ldots, N_f$ and $\ell = 1, 2, \ldots, K$ is understood modulo K.[2] Note that the link fields form *chiral* bifundamental multiplets of the gauge group.

The tree level superpotential has three types of terms serving different purposes,

$$W_{\mathrm{tree}} = W_{\mathrm{OR}} + W_{\mathrm{hop}} + W_{\mathrm{def}}\,. \tag{2.3}$$

The O'Raifeartaigh terms

$$W_{\mathrm{OR}} = \beta \sum_{\ell} s_{\ell} \times \left(\det(\Omega_{\ell}) - v^{N_c}\right) \tag{2.4}$$

— where the s_{ℓ} are singlet fields not shown in the quiver diagram — turn each bifundamental field Ω_{ℓ} into an $\mathrm{SL}(N_c, \mathbb{C})$ linear sigma model. This is important for deconstruction

[2]For deconstruction purposes one takes the $K \to \infty$ limit, but from the 4D point of view K is a fixed parameter of the field theory; in our analysis we shall assume K to be largish but finite.

purposes, and also allows us to model the Ω_ℓ with unitary matrices without worrying about the zero eigenvalues. The hopping superpotential

$$W_{\rm hop} = \gamma \sum_{\ell=1}^{K} \sum_{f=1}^{N_f} \left(\widetilde{Q}^f_{\ell+1} \Omega_\ell Q_{\ell,f} - \mu_f \, \widetilde{Q}^f_\ell Q_{\ell,f} \right) \tag{2.5}$$

describes quarks' masses and interactions which let them 'hop' between quiver nodes; in 5D terms, this allows quark propagation in the deconstructed x^4 direction. Finally, we have the deformation superpotential

$$W_{\rm def} = \mathrm{tr}\Big(\mathcal{W}(\Omega_K \Omega_{K-1} \cdots \Omega_2 \Omega_1) \Big) \equiv \sum_{p=1}^{d} \frac{g_p}{p} \, \mathrm{tr}\Big((\Omega_K \cdots \Omega_1)^p \Big) \tag{2.6}$$

for some polynomial $\mathcal{W}(X) = \sum_p \frac{g_p}{p} X^p$ with constant coefficients $g_1, \ldots, g_d$. The $W_{\rm def}$ deforms the 4D theory away from deconstructed SQCD_5, breaks the Coulomb branch of the moduli space into a discrete set of vacua, and allows formation of the gaugino condensates. This deformation is analogous to the tree-level superpotential for the adjoint field in [6, 14, 7] and is essential for understanding the on-shell chiral ring of the theory. It is also a key ingredient of the matrix model.

In the matrix model, the $[\mathrm{SU}(N_c)]^K$ gauge symmetry of the field theory becomes a $[\mathrm{U}(\hat{N})]^K$ global symmetry and we take the $\hat{N} \to \infty$ limit. Consequently, the bifundamental fields Ω_ℓ become $\hat{N} \times \hat{N}$ unitary matrices U_ℓ, and for chirality's sake we should integrate each matrix U_ℓ over the $\mathrm{U}(\hat{N})$ group manifold or an equivalent variety. However, in light of the O'Raifeartaigh terms (2.4) we restrict the determinants of the U_ℓ matrices and integrate them over the $\mathrm{SU}(\hat{N})$ group manifold. To accommodate the v^{N_c} factor, we rescale the field-matrix correspondence according to $\Omega_\ell \leftrightarrow \hat{v} \times \mathrm{U}_\ell$ where $\hat{v} = v +$ quantum corrections. Such quantum corrections are computable in the field theory — cf. section 4.3 of [12] for details — but in the matrix model they need to be put in by hand.

The quark sector of the field theory is non-chiral — for each quark Q_ℓ there is an antiquark $\widetilde{Q}_\ell$ with opposite quantum numbers. In the matrix model, the quarks become complex rectangular $\hat{N} \times \hat{F}$ matrices A_ℓ while the antiquark fields $\widetilde{Q}_\ell$ become conjugate $\hat{F} \times \hat{N}$ matrices $B_\ell = A^\dagger_\ell$. When we take the $\hat{N} \to \infty$ limit, we have two options for the flavor number $\hat{F}$ of the matrix model: we may keep it fixed (i.e., $\hat{F} \equiv N_f$), or we may let it grow while keeping the flavor/color ratio fixed, $\hat{F}/\hat{N} \equiv N_f/N_c$ [15]. In the 't Hooft limit of fixed $\hat{F}$, the bifundamental sector dominates the matrix model in the large color limit, and the quark sector becomes quenched — its backreaction on the bifundamental sector becomes negligible. This limit oversimplifies the physics but makes for a simple $1/\hat{N}$ perturbation theory in terms of Feynman-like diagrams' topology. On the other hand, in the un-quenched limit of $\hat{F}, \hat{N} \to \infty$ the matrix model has rich flavor physics, but the $1/\hat{N}$ expansion becomes much more difficult. Consequently, we use the 't Hooft limit in this article and leave the un-quenched flavor physics for future research.

Together, the U_ℓ, A_ℓ, and B_ℓ matrices comprise the entire matrix model. Its partition function is defined as the following matrix integral:

$$\mathcal{Z} = \mathcal{C}\prod_{\ell=1}^{K}\left\{\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_\ell]\int\!\!\int_{B_\ell=A_\ell^\dagger} dB_\ell\, dA_\ell\right\}\exp\left(-\frac{\hat{N}}{\hat{S}}\widehat{W}(\text{all }\ \mathrm{U}_\ell, B_\ell, A_\ell)\right) \tag{2.7}$$

where the matrix potential is

$$\begin{aligned}\widehat{W}(\mathrm{U}, B, A) &= \widehat{W}_{\mathrm{def}}(\mathrm{U}) + \widehat{W}_{\mathrm{hop}}(\mathrm{U}, B, A)\\ &= \sum_{p=1}^{d}\frac{g_p \hat{v}^{pK}}{p}\,\mathrm{tr}\Big((\mathrm{U}_K\,\mathrm{U}_{K-1}\cdots\mathrm{U}_2\,\mathrm{U}_1)^p\Big) + \\ &\quad +\gamma\sum_{\ell=1}^{K}\Big(\hat{v}\,\mathrm{tr}(B_{\ell+1}\,\mathrm{U}_\ell\,A_\ell) - \mathrm{tr}(\hat{\mu}A_\ell B_\ell)\Big)\end{aligned} \tag{2.8}$$

where $\hat{\mu}$ is an $\hat{F}\times\hat{F}$ matrix with eigenvalues $\mu_1,\ldots,\mu_{\hat{F}}$. Note that the matrix potential does not contain O'Raifeartaigh terms analogous to the W_{OR} of the field theory, but integrating the U_ℓ over the $\mathbf{S}\mathrm{U}(\hat{N})$ instead of $\mathrm{U}(\hat{N})$ has the same effect.

The denominator $\hat{S}$ in the exponent in eq. (2.7) is the overall coupling constant of the matrix model; as usual, $\hat{S}$ is fixed while $\hat{N}\to\infty$. Under gauge-matrix duality, $\hat{S}$ is dual to the net gaugino condensate $S=\sum_i S_i$ of all subgroups of the $\mathrm{SU}(N_c)_{\mathrm{diag}} = \mathrm{diag}[\text{all }\ \mathrm{SU}(N_c)_\ell]$. In the field theory, the S_i and hence the S emerge from the on-shell chiral ring, and then need to be integrated in to an effective off-shell superpotential, but in the matrix model the $\hat{S}$ is an input parameter and the effective superpotential emerges from a more direct calculation we shall perform in section 3.

Finally, let us fix the overall normalization factor $\mathcal{C}$ in eq. (2.7). Each U_ℓ is integrated over a compact manifold, while each $A_\ell = B_\ell^\dagger$ is integrated over the non-compact $\mathbb{C}^{\hat{N}\hat{F}}$, but the integral is gaussian. Hence, we let

$$\mathcal{C} = \frac{1}{\Big(\mathrm{Vol}[\mathrm{SU}(\hat{N})]\Big)^K}\times\frac{1}{\eta^{\hat{N}\hat{F}K}} \tag{2.9}$$

where η is the gaussian integral for a single quark mode of an average mass. The 'average' here does not have to be the arithmetic or the geometric mean, any representative value will do, and so we use $\gamma\hat{v}$ because the modes vary in mass from $\gamma(\hat{v}-\mu_f)$ to $\gamma(\hat{v}+\mu_f)$. Hence,

$$\eta = \int dx\, d\bar{x}\,\exp\left(-\frac{\hat{N}}{\hat{S}}\gamma\hat{v}\,|x|^2\right) = \frac{2\pi\hat{S}}{\hat{N}\gamma\hat{v}}\,. \tag{2.10}$$

2.2 Loop equations of the matrix model

Having defined our matrix model we now need to verify that it is indeed dual to the 4D gauge theory which deconstructs the SQCD_5. In this section we shall verify that the loop equations of the matrix model are similar to the anomaly equations of the gauge theory. To be precise, the matrix model is dual to a rather small part of the gauge theory — namely

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the subring of its chiral ring involving either the gaugino condensates or the mesons[3] — but the anomaly equations for that part of the gauge theory should be accurately reproduced by the loop equations of matrix model.

The loop equations follow from infinitesimal holomorphic changes of integration variables. For a simplified example, consider a toy model of a single unitary $\hat{N} \times \hat{N}$ matrix U,

$$\mathcal{Z} = \int_{\mathrm{U}(\hat{N})} d\omega[\mathrm{U}] \exp\left(-\frac{\hat{N}}{\hat{S}} \operatorname{tr}\Big(\widehat{W}(\mathrm{U})\Big)\right) \tag{2.11}$$

where $d\omega[\mathrm{U}]$ is the holomorphic form (1.2) of the Haar measure. Let us change

$$\mathrm{U} \mapsto \mathrm{U}' = \mathrm{U} + \epsilon \times f(\mathrm{U}) \tag{2.12}$$

where f is a holomorphic function of U. Generally, this breaks the unitarity of U, hence we should deform the integration variety from $\Gamma = \mathrm{U}(\hat{N})$ to Γ' which spans the U' for $\mathrm{U} \in \mathrm{U}(\hat{N})$. However, $\Gamma' \cong \Gamma$ and hence this deformation does not affect the holomorphic integral (2.11). On the other hand, the variable change (2.12) itself has a non-trivial jacobian

$$J \equiv \frac{d\omega[\mathrm{U}']}{d\omega[\mathrm{U}]} = 1 + \sum_{i,j} \frac{\delta(\mathrm{U}^{-1}\, d\,\mathrm{U})_{i,j}}{(\mathrm{U}^{-1}\, d\,\mathrm{U})_{i,j}} = 1 + \epsilon\left[\sum_{jk} \frac{\partial f_{kj}(\mathrm{U})}{\partial\, \mathrm{U}_{kj}} - \hat{N} \operatorname{tr}\Big(U^{-1} f(\mathrm{U})\Big)\right]. \tag{2.13}$$

Also, the integrand of the matrix integral changes according to

$$\exp\left(-\frac{\hat{N}}{\hat{S}} \operatorname{tr}\Big(\widehat{W}(\mathrm{U})\Big)\right) \mapsto \text{same} \times \left[1 - \epsilon \frac{\hat{N}}{\hat{S}} \operatorname{tr}\Big(\widehat{W}'(\mathrm{U}) f(\mathrm{U})\Big)\right]. \tag{2.14}$$

Altogether, we have changed the matrix integral by

$$\begin{aligned} \delta\mathcal{Z} &= \epsilon\mathcal{Z} \times \left\langle \left(\sum_{jk} \frac{\partial f_{kj}(\mathrm{U})}{\partial\, \mathrm{U}_{kj}} - \hat{N} \operatorname{tr}\Big(U^{-1} f(\mathrm{U})\Big) - \frac{\hat{N}}{\hat{S}} \operatorname{tr}\Big(\widehat{W}'(\mathrm{U}) f(\mathrm{U})\Big)\right) \right\rangle \\ &\equiv \epsilon \int_{\mathrm{U}(\hat{N})} d\omega[\mathrm{U}] \exp\left(-\frac{\hat{N}}{\hat{S}} \operatorname{tr}\Big(\widehat{W}(\mathrm{U})\Big)\right) \times \Big[\ \bullet\ \Big]. \end{aligned} \tag{2.15}$$

On the other hand, we have done nothing but changed the integration variable from U to U', hence the integral should not change at all. Therefore, for any holomorphic matrix $\mapsto$ matrix function $f(\mathrm{U})$ we must have

$$\left\langle \sum_{jk} \frac{\partial f_{kj}(\mathrm{U})}{\partial\, \mathrm{U}_{kj}} - \hat{N} \operatorname{tr}\Big(U^{-1} f(\mathrm{U})\Big) - \frac{\hat{N}}{\hat{S}} \operatorname{tr}\Big(\widehat{W}'(\mathrm{U}) f(\mathrm{U})\Big) \right\rangle = 0\,. \tag{2.16}$$

[3]The matrix models also has analogues of the baryonic and antibaryonic operators of the gauge theory, but such "dual baryons" are not matrices and are rather difficult to handle. For this reason, we shall limit our analysis here to the duals of the gaugino condensates and mesons only.

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This toy example shows how to derive loop equations for unitary matrix models. Let us apply this technology to the unitary U_ℓ matrices of our big matrix integral (2.7). Let us pick one matrix, say U_ℓ and change

$$U_\ell \mapsto U'_\ell = \mathrm{U}_\ell + \epsilon \times f(\mathrm{U}_\ell, \ldots) \tag{2.17}$$

where f is a holomorphic function of the U_ℓ and other matrices of the model (denoted by the $\ldots$). To preserve the symmetries of the model, $f(\mathrm{U}_\ell, \ldots)$ must be covariant, *i.e.* transform like U_ℓ under the $[\mathrm{U}(\hat{N})]^K$. Also, because the U_ℓ is integrated over the $\mathbf{S}\mathrm{U}(\hat{N})$ group manifold rather than $\mathrm{U}(\hat{N})$, we must preserve the $\det(\mathrm{U}'_\ell) = \det(\mathrm{U}_\ell) = 1$ condition, hence f must satisfy

$$\mathrm{tr}\Big(\mathrm{U}_\ell^{-1}\, f(\mathrm{U}_\ell, \ldots)\Big) = 0\,. \tag{2.18}$$

On the other hand, we do not need to preserve the unitarity of the U'_ℓ because we may deform the integration variety from $\Gamma = \mathrm{SU}(\hat{N})$ to a nearby $\Gamma' \cong \Gamma$. Hence, proceeding exactly as in the toy example above, we find that for any covariant holomorphic f which satisfies eq. (2.18) we must have

$$\left\langle \sum_{jk} \frac{\partial f_{kj}(\mathrm{U}_\ell, \ldots)}{\partial(\mathrm{U}_\ell)_{kj}} - \frac{\hat{N}}{\hat{S}}\,\mathrm{tr}\Big(\frac{\partial \widehat{W}(\mathrm{U}, A, B)}{\partial\, \mathrm{U}_\ell}\, f(\mathrm{U}_\ell, \ldots)\Big) \right\rangle = 0\,. \tag{2.19}$$

Now let us focus on functions f which depend only on the unitary link matrices but not on the quark matrices. By covariance, products of the $\mathrm{U}_{\ell'}$ must be taken in the order of the quiver, hence $f(\mathrm{U}_\ell, \text{other } \mathrm{U}_{\ell'})$ must be a linear combination of $(\mathrm{U}_\ell\, \mathrm{U}_{\ell-1} \cdots \mathrm{U}_1 \times \mathrm{U}_K \cdots \mathrm{U}_{\ell+1})^p\, \mathrm{U}_\ell$ for $p = 0, 1, 2, \ldots$, which can be summarized in a power series

$$\begin{aligned} \tilde{f}(X) &= \sum_{p=0}^{\infty} X^{-1-p} \times \Big(\mathrm{U}_\ell\, \mathrm{U}_{\ell-1} \cdots \mathrm{U}_1\, \mathrm{U}_K \cdots \mathrm{U}_{\ell+1}\Big)^p \times \mathrm{U}_\ell \\ &= \frac{1}{X - \mathrm{U}_\ell\, \mathrm{U}_{\ell-1} \cdots \mathrm{U}_1\, \mathrm{U}_K \cdots \mathrm{U}_{\ell+1}} \times \mathrm{U}_\ell \end{aligned} \tag{2.20}$$

in an auxiliary complex variable X. In field theory we use this series as it is, but in the matrix model we must correct for the trace condition (2.18), thus

$$f(\mathrm{U}_\ell, \ldots) = \left(\left[\frac{1}{X - \mathrm{U}_\ell\, \mathrm{U}_{\ell-1} \cdots \mathrm{U}_1\, \mathrm{U}_K \cdots \mathrm{U}_{\ell+1}} - \frac{1}{\hat{S}}\, R(X)\right]\right) \times \mathrm{U}_\ell \tag{2.21}$$

where the second term inside the brackets assures $\mathrm{tr}[\cdot] = 0$. Specifically,

$$R(X) \stackrel{\text{def}}{=} \frac{\hat{S}}{\hat{N}} \times \mathrm{tr}\left(\frac{1}{X - \mathrm{U}_\ell\, \mathrm{U}_{\ell-1} \cdots \mathrm{U}_1\, \mathrm{U}_K \cdots \mathrm{U}_{\ell+1}}\right) \tag{2.22}$$

which is same for all ℓ and remains finite in the $\hat{N} \to \infty$ limit. For f as in eq. (2.21)

$$\sum_{jk} \frac{\partial f_{kj}(\mathrm{U}_\ell, \ldots)}{\partial(\mathrm{U}_\ell)_{kj}} = \frac{\hat{N}^2}{\hat{S}^2}\left[X R^2(X) - \hat{S} R(X) + \frac{\hat{S}}{\hat{N}^2}\,(X R'(X) + R(X))\right] \tag{2.23}$$

while

$$\operatorname{tr}\left(\frac{\partial\widehat{W}_{\rm def}}{\partial\,\mathrm{U}_\ell}\,f(\mathrm{U}_\ell,\ldots)\right) = \frac{\hat{N}}{\hat{S}}\left[X\mathcal{W}'(X)R(X) - \hat{S}\mathcal{W}'(X) + \hat{S}P(X) - CR(X)\right] \qquad (2.24)$$

where

$$\mathcal{W}'(X) = \sum_{p=1}^{d} g_p\hat{v}^{pK}X^{p-1}\,, \qquad (2.25)$$

$$P(X) = \frac{1}{\hat{N}}\operatorname{tr}\left((\mathrm{U}_K\cdots\mathrm{U}_1)\frac{\mathcal{W}'(X)-\mathcal{W}'(\mathrm{U}_K\cdots\mathrm{U}_1)}{X-\mathrm{U}_K\cdots\mathrm{U}_1}\right)$$

$$\text{(a polynomial of } X \text{ of degree} = d-2) \qquad (2.26)$$

$$\text{and} \qquad C = \frac{1}{\hat{N}}\operatorname{tr}\Big((\mathrm{U}_K\cdots\mathrm{U}_1)\mathcal{W}'(\mathrm{U}_K\cdots\mathrm{U}_1)\Big)\,. \qquad (2.27)$$

Note that in the large $\hat{N}$ limit C and $P(X)$ remain finite, hence the right hand side of eq. (2.24) grows like $\hat{N}$. By comparison,

$$\operatorname{tr}\left(\frac{\partial\widehat{W}_{\rm hop}}{\partial\,\mathrm{U}_\ell}\,f(\mathrm{U}_\ell,\ldots)\right) = O(\hat{F})\,, \qquad (2.28)$$

which is much smaller than $O(\hat{N})$ in the 't Hooft limit. Therefore, eq. (2.19) becomes

$$\Big\langle XR^2(X) - (X\mathcal{W}'(X)+\hat{S}-C)\times R(X) + \hat{S}\mathcal{W}'(X) - \hat{S}P(X)\Big\rangle = O\left(\frac{1}{\hat{N}}\right) \approx 0\,. \qquad (2.29)$$

Moreover, for $\hat{N}\to\infty$ matrix averages factorize as

$$\langle R^2(X)\rangle \to \langle R(X)\rangle^2\,, \qquad \langle C\times R(X)\rangle \to \langle C\rangle\times\langle R(X)\rangle\,, \qquad (2.30)$$

and this gives us a loop equation for the $\langle R(X)\rangle$, namely

$$X\,\langle R(X)\rangle^2 - \Big(X\mathcal{W}'(X)+\hat{S}-\langle C\rangle\Big)\times\langle R(X)\rangle + \hat{S}\mathcal{W}'(X) + \hat{S}\,\langle P(X)\rangle = 0\,. \qquad (2.31)$$

On the field theory side, we have a similar quadratic equation for the gaugino condensate resolvent

$$R(X) = \operatorname{tr}\left(\frac{(W^\alpha W_\alpha)_\ell}{32\pi^2}\times\frac{1}{X-\Omega_\ell\Omega_{\ell-1}\cdots\Omega_1\Omega_K\cdots\Omega_{\ell+1}}\right) \qquad \text{(same for all } \ell)\,. \qquad (2.32)$$

In the on-shell chiral ring, this resolvent satisfies

$$XR^2(X) - \widetilde{\mathcal{W}}(X)R(X) + F(X) = 0 \qquad (2.33)$$

where

$$\widetilde{\mathcal{W}}(X) = \sum_{p=1}^{d} g_pX^p + \beta v^{N_c}\langle s\rangle \qquad \text{(same } \langle s_\ell\rangle\ \forall\ell) \qquad (2.34)$$

and $F(X)$ is another polynomial (of degree $< d$) which depends on vacuum state of the field theory. Eqs. (2.31) and (2.33) are obviously dual to each other; to make them identical we simply need to identify

$$X^{\rm field}_{\rm theory} \longleftrightarrow \hat{v}^K \times X^{\rm matrix}_{\rm model}\,, \quad R(X)^{\rm field}_{\rm theory} \longleftrightarrow \hat{v}^{-K} \times \langle R(X)\rangle^{\rm matrix}_{\rm model}\,, \tag{2.35}$$

$$(\beta v^{N_c}\langle s\rangle)^{\rm field}_{\rm theory} \longleftrightarrow (\hat{S} - \langle C\rangle)^{\rm matrix}_{\rm model}\,, \quad F(X)^{\rm field}_{\rm theory} \longleftrightarrow \hat{v}^{-K} \times \hat{S}\left(\mathcal{W}'(X) + \langle P(X)\rangle\right)^{\rm matrix}_{\rm model}. \tag{2.36}$$

This correspondence explains why $\hat{S}$ is dual to the net gaugino condensate

$$S_{\rm net} = \frac{\langle {\rm tr}(W^\alpha W_\alpha)\rangle}{32\pi^2} = \lim_{X\to\infty}(XR(X))\,. \tag{2.37}$$

Solving the anomaly equation (2.33), we have

$$R(X) = \frac{\widetilde{\mathcal{W}}(X) - \sqrt{\widetilde{\mathcal{W}}^2(X) - 4XF(X)}}{2X} \xrightarrow[X\to\infty]{} \frac{F(X)}{\widetilde{\mathcal{W}}(X)}\,. \tag{2.38}$$

According to eq. (2.36), the matrix dual of the right hand side is

$$\hat{v}^{-K}\hat{S} \times \frac{\mathcal{W}'(X) + \langle(P(X)\rangle}{X\mathcal{W}'(X) + {\rm const.}} \xrightarrow[X\to\infty]{} \hat{S} \times \frac{\hat{v}^{-K}}{X^{\rm matrix}_{\rm model}} = \frac{\hat{S}}{X^{\rm field}_{\rm theory}} \tag{2.39}$$

(note that $P(X)$ has lower degree in X than $\mathcal{W}'(X)$), and therefore

$$(S_{\rm net})^{\rm field}_{\rm theory} \longleftrightarrow (\hat{S})^{\rm matrix}_{\rm model}\,. \tag{2.40}$$

The mesonic resolvents of the gauge theory also have matrix duals. On the field theory side, chiral mesonic operators with quarks and antiquarks at different quiver nodes are packaged into a bunch of resolvents

$$\mathcal{M}_{\ell',\ell}(X) = \widetilde{Q}_{\ell'}\Omega_{\ell'-1}\cdots\Omega_\ell\,\frac{1}{X - \Omega_{\ell-1}\cdots\Omega_1\Omega_K\cdots\Omega_\ell}\,Q_\ell \tag{2.41}$$

subject to periodicity conditions

$$-\mathcal{M}_{\ell'=\ell+K} + X\mathcal{M}_{\ell'=\ell} = M_\ell \equiv (\widetilde{Q}_\ell Q_\ell) \tag{2.42}$$

where the right hand side is an ordinary mesonic operator which does not depend on X. Besides their quiver indices, the mesonic resolvents are also $N_f \times N_f$ matrices in the flavor space. On the matrix-model side, they are dual to (the averages of) $\hat{F} \times \hat{F}$ matrices

$$\mathbf{M}_{\ell',\ell}(X) = B_{\ell'}\,{\rm U}_{\ell'-1}\cdots{\rm U}_\ell\,\frac{1}{X - {\rm U}_{\ell-1}\cdots{\rm U}_1\,{\rm U}_K\cdots{\rm U}_\ell}\,A_\ell \tag{2.43}$$

which satisfy a similar periodicity relation

$$-\mathbf{M}_{\ell'=\ell+K}(X) + X\mathbf{M}_{\ell'=\ell}(X) = M_\ell \equiv (B_\ell A_\ell)\,. \tag{2.44}$$

Because of the $\hat{v}$ factors in the $\Omega_\ell \leftrightarrow \mathrm{U}_\ell \times \hat{v}$ correspondence, in the mesonic sector we expect the gauge-matrix duality to work according to

$$X^{\text{field}}_{\text{theory}} \leftrightarrow \hat{v}^K \times X^{\text{matrix}}_{\text{model}}\,, \qquad \mathcal{M}_{\ell',\ell}(X)^{\text{field}}_{\text{theory}} \leftrightarrow \hat{v}^{\ell'-\ell-K} \times \left\langle \mathbf{M}_{\ell',\ell}(X) \right\rangle^{\text{matrix}}_{\text{model}}\,, \tag{2.45}$$

To derive loop equations for the mesonic resolvents of the matrix model, we need to vary the A_ℓ and the B_ℓ matrices independently of each other. Note that the conjugacy constraint $B_\ell = A_\ell^\dagger$ is just a special case of integrating the $2\hat{N}\hat{F}$ complex numbers comprising each (A_ℓ, B_ℓ) over a variety Γ of real dimension $2\hat{N}\hat{F}$, we "just happened" to choose $\Gamma = \{B_\ell = A_\ell^\dagger\}$. But as long as the integrand is a holomorphic function of both A_ℓ and B_ℓ (views as independent variables), we may deform the integration variety without changing the integral,

$$\iint\limits_{B_\ell = A_\ell^\dagger} dA_\ell\, dB_\ell\, f(A_\ell, B_\ell, \ldots) = \iint\limits_{\Gamma'} dA_\ell\, dB_\ell\, f(A_\ell, B_\ell, \ldots) \qquad \text{for } \Gamma' \cong \{B_\ell = A_\ell^\dagger\}\,. \tag{2.46}$$

Therefore, *small* variations of the A_ℓ and the B_ℓ matrices don't need to be conjugate to each other — the discrepancy will deform the integration variety a bit, but small deformations do not affect the integral.

Minding this rule, let us vary any one B_ℓ matrix while the "conjugate" A_ℓ matrix remains unchanged. Specifically, let

$$B_\ell \mapsto B'_\ell = B_\ell + \epsilon \times B_{\ell'} \times \frac{\mathrm{U}_{\ell'-1} \cdots \mathrm{U}_\ell}{X - \mathrm{U}_{\ell-1} \cdots \mathrm{U}_\ell} \tag{2.47}$$

where ϵ is an infinitesimal $\hat{F} \times \hat{F}$ matrix in the flavor space. The jacobian for such change of variables is

$$\frac{d^{\hat{F}\hat{N}} B'_\ell}{d^{\hat{F}\hat{N}} B_\ell} = 1 + \delta_{\ell',\ell} \times \operatorname{tr}(\epsilon) \times \operatorname{tr}\left(\frac{1}{X - \mathrm{U}_{\ell-1} \cdots \mathrm{U}_\ell}\right) = 1 + \delta_{\ell',\ell} \times \operatorname{tr}(\epsilon) \times \frac{\hat{N}}{\hat{S}} R(X)\,, \tag{2.48}$$

while the potential changes by

$$\delta\widehat{W} = \delta\widehat{W}_{\text{hop}} = \gamma\hat{v} \operatorname{tr}\left(\epsilon \times \mathbf{M}_{\ell',\ell-1}(X)\right) - \gamma \operatorname{tr}\left(\epsilon \times \mathbf{M}_{\ell',\ell}(X) \times \hat{\mu}\right). \tag{2.49}$$

But altogether, this is just a change of an integration variable which does not change the integral at all, hence

$$\begin{aligned} 0 &= \langle \delta(\text{Jacobian}) \rangle - \frac{\hat{N}}{\hat{S}} \left\langle \delta\widehat{W} \right\rangle \\ &= \delta_{\ell',\ell} \times \operatorname{tr}(\epsilon) \times \frac{\hat{N}}{\hat{S}} \langle R(X) \rangle - \frac{\hat{N}}{\hat{S}} \gamma \operatorname{tr}\left(\epsilon \times \left(\hat{v}\left\langle \mathbf{M}_{\ell',\ell-1}(X)\right\rangle - \mathbf{M}_{\ell',\ell}(X) \times \hat{\mu}\right)\right). \end{aligned} \tag{2.50}$$

This must hold true for any ϵ matrix, therefore

$$\hat{v}\left\langle \mathbf{M}_{\ell',\ell-1}(X)\right\rangle - \left\langle \mathbf{M}_{\ell',\ell}(X)\right\rangle \times \hat{\mu} = \delta_{\ell,\ell'} \times \gamma^{-1} \langle R(X) \rangle \times \mathbf{1}_{\hat{F}\times\hat{F}}\,. \tag{2.51}$$

Likewise, changing the A_ℓ matrix by

$$\delta A_\ell = \frac{\mathrm{U}_{\ell-1}\cdots\mathrm{U}_{\ell'}}{X-\mathrm{U}_{\ell'-1}\cdots\mathrm{U}_{\ell'}} \times A_{\ell'} \times \epsilon \tag{2.52}$$

while the B_ℓ matrix remains unchanged leads to another loop equation, namely

$$\hat{\upsilon}\,\big\langle \mathbf{M}_{\ell+1,\ell'}(X)\big\rangle - \hat{\mu}\times\big\langle \mathbf{M}_{\ell,\ell'}(X)\big\rangle = \delta_{\ell,\ell'}\times\gamma^{-1}\,\langle R(X)\rangle \times \mathbf{1}_{\hat{F}\times\hat{F}}\,. \tag{2.53}$$

Both loop equations (2.51) and (2.53) look exactly like their field theory counterparts:

$$\mathcal{M}_{\ell',\ell-1}(X) - \mathcal{M}_{\ell',\ell}(X)\times\mu = \mathcal{M}_{\ell'+1,\ell}(X) - \mu\times\mathcal{M}_{\ell',\ell}(X) = \delta_{\ell,\ell'}\times\gamma^{-1}\,R(X)\times\mathbf{1}_{N_f\times N_f}\,. \tag{2.54}$$

The only difference is in the $\hat{\upsilon}$ factors — and that is in accordance with the correspondence rules (2.45) and (2.35).

Having two sets (2.51) and (2.53) of mesonic loop equations is *not* redundant. Combining both sets with the periodicity equations (2.44) allows us to solve for all the mesonic resolvents (or rather their averages) in terms of a single matrix $\langle M\rangle$. The solution works exactly as for the gauge theory, so instead of copying from [12] almost verbatim, let us simply state the result:

$$\langle \mathbf{M}_{\ell'=\ell}(X)\rangle = \frac{\langle M\rangle + \hat{\upsilon}^{-K}\mu^{K-1}\gamma^{-1}\,\langle R(X)\rangle\times\mathbf{1}}{X-\hat{\upsilon}^{-K}\hat{\mu}^{K}}\,, \tag{2.55}$$

$$\langle \mathbf{M}_{\ell'>\ell}(X)\rangle = \frac{\hat{\mu}^{\ell'-\ell-1}}{\hat{\upsilon}^{\ell'-\ell}} \times \frac{\hat{\mu}\times\langle M\rangle + \gamma^{-1}X\,\langle R(X)\rangle\times\mathbf{1}}{X-\hat{\upsilon}^{-K}\hat{\mu}^{K}}\,, \tag{2.56}$$

where $\langle M\rangle = \langle B_\ell A_\ell\rangle$ must be same for all ℓ and must commute with the quark mass matrix $\hat{\mu}$.[4]

Finally, there is yet another bunch of loop equations for the mesonic resolvents stemming from A and B dependent variations of the U matrices. Let us pick any one U_ℓ and vary it according to

$$\delta\,\mathrm{U}_\ell = \left(A_{\ell+1}\times\epsilon\times B_\ell\times\frac{1}{X-\mathrm{U}_{\ell-1}\cdots\mathrm{U}_1\,\mathrm{U}_K\cdots\mathrm{U}_\ell}\right) - \frac{\mathrm{tr}\Big(\mathrm{U}_\ell^{-1}\times(\circ)\Big)}{\hat{N}}\times\mathrm{U}_\ell \tag{2.57}$$

where the second term assures $\delta\det(\mathrm{U}_\ell)=0$. The trace in the second term evaluates to

$$\mathrm{tr}\left(\epsilon\times\frac{\mathbf{M}_{\ell+N,\ell+1}(X)-\mathbf{M}_{\ell+N,\ell+1}(0)}{X}\right). \tag{2.58}$$

The jacobian of the variable change (2.57) is given by

$$\delta J = \sum_{jk}\frac{\partial(\delta\,\mathrm{U}_\ell)_{jk}}{\partial(\mathrm{U}_\ell)_{jk}} = \frac{\hat{N}}{\hat{S}}\,\mathrm{tr}\left(\epsilon\times\begin{bmatrix} R(X)\times\mathbf{M}_{\ell+K,\ell+1}(X) \\ \quad -\hat{S}\times\dfrac{\mathbf{M}_{\ell+K,\ell+1}(X)-\mathbf{M}_{\ell+K,\ell+1}(0)}{X} \\ \quad +O\left(\dfrac{1}{\hat{N}^2}\right)\end{bmatrix}\right), \tag{2.59}$$

[4]Strictly speaking, $\langle M\rangle$ must commute with the $\hat{\mu}^K$ matrix rather than with the $\hat{\mu}$ itself. Also, the matrix elements of $\langle B_\ell A_\ell\rangle$ which commute with $\hat{\mu}^K$ but not with $\hat{\mu}$ have ℓ-dependent phases. To avoid this mess, we assume that all distinct mass eigenvalues also have distinct $K^{\underline{\text{th}}}$ powers.

while the potential varies according to

$$\delta\widehat{W}_{\rm hop} = \gamma\hat{v}\,{\rm tr}\Big(\epsilon \times \mathbf{M}_{\ell,\ell}(X) \times M_{\ell+1}\Big) + O\bigg(\frac{1}{\hat{N}}\bigg), \tag{2.60}$$

$$\delta\widehat{W}_{\rm def} = = {\rm tr}\left(\epsilon \times \begin{bmatrix} \mathcal{W}'(X) \times \mathbf{M}_{\ell+K,\ell+1}(X) \\ \quad -C \times \dfrac{\mathbf{M}_{\ell+K,\ell+1}(X) - \mathbf{M}_{\ell+K,\ell+1}(0)}{X} \\ \quad -t_\ell(X) \end{bmatrix}\right) \tag{2.61}$$

where C is as in eq. (2.27) and $t_\ell(X)$ is an $\hat{F} \times \hat{F}$ matrix-valued polynomial of X whose actual form does not affect the following argument. Altogether, these changes must cancel out of the matrix integral, hence demanding

$$\langle \delta J \rangle - \frac{\hat{N}}{\hat{S}} \left\langle \delta\widehat{W} \right\rangle = 0 \qquad \forall\epsilon \tag{2.62}$$

we arrive at

$$\begin{aligned} &\gamma\hat{v}\,\langle \mathbf{M}_{\ell,\ell}(X) \times M_{\ell+1} \rangle - \langle R(X) \times \mathbf{M}_{\ell+K,\ell+1}(X) \rangle = \\ &\quad = \left[\mathcal{W}'(X) + \frac{\hat{S}-C}{X}\right] \times \langle \mathbf{M}_{\ell+K,\ell+1}(X) \rangle - \frac{\hat{S}-C}{X} \times \langle \mathbf{M}_{\ell+K,\ell+1}(0) \rangle \\ &\quad = \langle t_\ell(X) \rangle + O\bigg(\frac{1}{\hat{N}}\bigg). \end{aligned} \tag{2.63}$$

In the large $\hat{N}$ limit matrix averages factorize, and this gives us another family of loop equations, namely

$$\begin{aligned} &\gamma\hat{v}\,\langle \mathbf{M}_{\ell,\ell}(X) \rangle \times \langle M_{\ell+1} \rangle + \left[\mathcal{W}'(X) + \frac{\hat{S}-C}{X} - \langle R(X) \rangle\right] \times \\ &\qquad \times \langle \mathbf{M}_{\ell+K,\ell+1}(X) \rangle - \frac{\hat{S}-C}{X} \times \langle \mathbf{M}_{\ell+K,\ell+1}(0) \rangle = \text{a polynomial of } X\,. \end{aligned} \tag{2.64}$$

Again, these loop equations are exactly similar to their field theory counterparts and we may solve them in a similar way. Making use of eqs. (2.55)–(2.56) and eq. (2.31), we transform eq. (2.64) into a quadratic equation for the mesonic "VEV" $\langle M \rangle$, namely

$$\begin{aligned} &(\gamma\hat{\mu}\,\langle M \rangle)^2 + \frac{\hat{\mu}^K}{X\hat{v}^K}\,[X\mathcal{W}'(X) + \hat{S} - C] \times (\gamma\hat{\mu}\,\langle M \rangle) + \frac{\hat{\mu}^K}{X\hat{v}^K}\,F(X) = \\ &\qquad\qquad = \left[1 - \frac{\hat{\mu}^K}{X\hat{v}^K}\right] \times (\text{a polynomial of } X)\,. \end{aligned} \tag{2.65}$$

Also, the $\langle M \rangle$ matrix is block-diagonal in the eigenbasis of $\hat{\mu}$ and does not depend on X. Hence, for each block we may substitute a different value of X into eq. (2.65) and apply the resulting equation to the block in question; our choice is $X = \hat{\mu}_f^K/\hat{v}^K$ which kills the right hand side of eq. (2.65) regardless of the matrix-valued polynomial we didn't spell out. Consequently, each block — and hence the whole matrix — satisfies

$$(\gamma\hat{\mu}\,\langle M \rangle)^2 + \left(\left(\frac{\hat{\mu}}{\hat{v}}\right)^K \mathcal{W}'\left(\left(\frac{\hat{\mu}}{\hat{v}}\right)^K\right) + \hat{S} - C\right) \times (\gamma\hat{\mu}\,\langle M \rangle) + F\left(\left(\frac{\hat{\mu}}{\hat{v}}\right)^K\right) = 0\,. \tag{2.66}$$

Note similarity between this equation and the loop equation (2.31); this allows us to write

$$\langle M_\ell \rangle = -\frac{X \langle R(X) \rangle^{\pm}}{\gamma \hat{\mu}} \qquad \text{evaluated for } X = \left(\frac{\hat{\mu}}{\hat{v}}\right)^K \tag{2.67}$$

where $\langle R(X) \rangle^{\pm}$ on the right hand side indicates the two solutions of eq. (2.31).

This completes our analysis of the loop equations of our matrix model. Having seen that those equations are dual to the anomaly equations of the gauge theory of the quiver diagram (1.3), we can be positive that our model is indeed dual to that gauge theory.

3. Calculating the matrix integral

In this section we evaluate the matrix integral (2.7) in a sequence of simple steps. First we integrate over the A_ℓ and B_ℓ matrices dual to the quarks and the antiquarks. Second, we reduce the integral over K link matrices U_ℓ to an integral over a single unitary matrix $\mathcal{U} = \mathrm{U}_K \, \mathrm{U}_{K-1} \cdots \mathrm{U}_1$. After that we follow the Dijkgraaf-Vafa method adapted to a unitary rather than hermitian matrix: we reduce the integral over the whole matrix $\mathcal{U}$ to an integral over its eigenvalues, and then we use the saddle-point approximation in the large $\hat{N}$ limit. In this limit, the eigenvalue spectrum becomes continuous with density $\rho(\lambda)$, the free energy has a $1/\hat{N}$ expansion where the sphere-level and the disk-level terms are given by spectral integrals, — and we relate the whole shmeer to the loop equation (2.31) and the period integrals of its Riemann surface.

Let us integrate over the quark matrices. The hopping potential (2.8) is bilinear with respect to the matrix elements of the A_ℓ and B_ℓ:

$$\widehat{W}_{\text{hop}} = \sum_{\ell,\ell'} \sum_{i,i'} \sum_{f,f'} B_{\ell',i',f'} \, \mathcal{D}^{\ell',i',f'}_{\ell,i,f} \, A^{\ell,i,f} \tag{3.1}$$

where

$$\mathcal{D}^{\ell',i',f'}_{\ell,i,f} = \hat{v} \delta^{\ell'}_{\ell+1} \, (\mathrm{U}_\ell)^{i'}_i \, \delta^{f'}_f - \delta^{\ell'}_\ell \, \delta^{i'}_i \, \hat{\mu}^{f'}_f \, . \tag{3.2}$$

This makes the quark-matrix integral a gaussian integral over $K\hat{N}\hat{F}$ independent complex variables, which we evaluate as

$$\prod_{\ell=1}^{K} \left\{ \iint\limits_{B_\ell = A_\ell^\dagger} d^{\hat{N}\hat{F}} A_\ell \, d^{\hat{N}\hat{F}} B_\ell \right\} \exp\left(-\frac{\hat{N}}{\hat{S}} \, \widehat{W}(\mathrm{U}, A, B) \right) = \left(\frac{2\pi \hat{S}}{\hat{N}\gamma} = \eta \hat{v} \right)^{K\hat{N}\hat{F}} \times \frac{1}{\operatorname{Det} \mathcal{D}} \tag{3.3}$$

where the determinant involves all indices: quiver, color, and flavor. In block form

$$\mathcal{D} = \begin{pmatrix} -\hat{\mu} & 0 & 0 & \cdots & \hat{v}\, \mathrm{U}_K \\ \hat{v}\, \mathrm{U}_1 & -\hat{\mu} & 0 & \cdots & 0 \\ 0 & \hat{v}\, \mathrm{U}_2 & -\hat{\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{v}\, \mathrm{U}_{K-1} & -\hat{\mu} \end{pmatrix}, \tag{3.4}$$

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and hence

$$\operatorname*{Det}_{K\hat{N}\hat{F}\times K\hat{N}\hat{F}}(\mathcal{D}) = \operatorname*{Det}_{\hat{N}\hat{F}\times\hat{N}\hat{F}}\left(\hat{v}^K\,\mathrm{U}_K\,\mathrm{U}_{K-1}\cdots\mathrm{U}_1 - \hat{\mu}^K\right) = \prod_{f=1}^{\hat{F}}\operatorname*{det}_{\hat{N}\times\hat{N}}\left(\hat{v}^K\,\mathrm{U}_K\,\mathrm{U}_{K-1}\cdots\mathrm{U}_1 - \hat{\mu}_f^K\right) \tag{3.5}$$

where on the right hand side the $\hat{\mu}_f$ are eigenvalues of the quark mass matrix $\hat{\mu}$.

Note that the 'quark' integral (3.3) depends on the link matrices U_ℓ only through their product $\mathcal{U} \equiv \mathrm{U}_K\,\mathrm{U}_{K-1}\cdots\mathrm{U}_1$. Likewise, the deformation superpotential $\mathcal{W}_{\mathrm{def}}$ depends on the U_ℓ only through their product. Hence, at this stage, we may write the matrix integral (2.7) as

$$\mathcal{Z} = \frac{1}{\left(\mathrm{Vol}[\mathrm{SU}(\hat{N})]\right)^K}\prod_{\ell=1}^{K}\left\{\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_\ell]\right\} f(\mathcal{U} \equiv \mathrm{U}_K\,\mathrm{U}_{K-1}\cdots\mathrm{U}_1) \tag{3.6}$$

where

$$f(\mathcal{U}) = \frac{\exp\left(-\frac{\hat{N}}{\hat{S}}\widehat{W}_{\mathrm{def}}(\mathcal{U})\right)}{\prod_{f=1}^{\hat{F}}\det_{\hat{N}\times\hat{N}}\left(\mathcal{U} - (\hat{\mu}_f/\hat{v})^K\right)}\,, \tag{3.7}$$

and $d\omega[\mathrm{U}_\ell]$ is the Haar measure for $\mathrm{U}_\ell \in \mathrm{SU}(\hat{N})$. This measure is left-invariant — for any *fixed* $V \in \mathrm{SU}(\hat{N})$, $d\omega[V\,\mathrm{U}] = d\omega[\mathrm{U}]$ — and this makes it easy to change variables in the unitary matrix integrals. In particular, in an integral over K matrices such as (3.6) we can set $\mathrm{U}_1 = V\,\mathrm{U}'_1$ (for $\ell = 1$ only) and have

$$d\omega[\mathrm{U}_K]\times\cdots\times d\omega[\mathrm{U}_2]\times d\omega[\mathrm{U}_1 = V\,\mathrm{U}'_1] = d\omega[\mathrm{U}_K]\times\cdots\times d\omega[\mathrm{U}_2]\times d\omega[\mathrm{U}'_1] \tag{3.8}$$

for any $V \in \mathrm{SU}(\hat{N})$ which does not depend on the U'_1 matrix, even if V depends on the *other* unitary matrices $\mathrm{U}_2,\ldots,\mathrm{U}_N$. Therefore

$$\begin{aligned}
&\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_K]\cdots\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_1] f(\mathrm{U}_K\cdots\mathrm{U}_1) = \\
&\qquad = \int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_K]\cdots\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_2]\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}'_1] f(\mathrm{U}_K\cdots\mathrm{U}_2\times V\,\mathrm{U}'_1) \\
&\quad \langle\!\langle \text{setting } V = (\mathrm{U}_K\cdots\mathrm{U}_2)^{-1} \Longrightarrow \mathrm{U}'_1 = V^{-1}\,\mathrm{U}_1 = \mathrm{U}_K\cdots\mathrm{U}_2\,\mathrm{U}_1 \equiv \mathcal{U}\,\rangle\!\rangle \\
&\qquad = \int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_K]\,\cdots\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}_2]\int_{\mathrm{SU}(\hat{N})} d\omega[\mathrm{U}'_1 = \mathcal{U}]\, f(\mathcal{U}) \\
&\qquad = \left(\mathrm{Vol}[\mathrm{SU}(\hat{N})]\right)^{K-1}\times\int_{\mathrm{SU}(\hat{N})} d\omega[\mathcal{U}]\, f(\mathcal{U}),
\end{aligned} \tag{3.9}$$

and hence

$$\mathcal{Z} = \frac{1}{\mathrm{Vol}[\mathrm{SU}(\hat{N})]}\times\int_{\mathrm{SU}(\hat{N})} d\omega[\mathcal{U}] f(\mathcal{U})\,. \tag{3.10}$$

Furthermore, according to eq. (3.7), $f(\mathcal{U})$ depends only on the eigenvalues of the unitary matrix $\mathcal{U}$ but not on its eigenvectors. Consequently, we decompose the $\hat{N}^2 - 1$ coordinates of the $\mathrm{SU}(\hat{N})$ group manifold into $\hat{N} - 1$ independent eigenvalues $e^{i\lambda_i}$ ($i =$

$1,\ldots,\hat{N}$ but $\sum_i \lambda_i \equiv 0 \bmod 2\pi$) and $\hat{N}(\hat{N}-1)$ angular variables $\theta_{i\neq j}$ describing the eigenvectors. The jacobian of this decomposition is given by

$$\frac{d\omega}{d(\lambda_i,\theta_{i,j})} = \text{const.} \times \prod_{i<j} 4\sin^2\frac{\lambda_i-\lambda_j}{2} \tag{3.11}$$

— the unitary version of the Vandermonde determinant — which depends only on the eigenvalues λ_i, hence integrating over the eigenvector variables $\theta_{i,j}$ we arrive at

$$\mathcal{Z} = \frac{\hat{v}^{K\hat{N}\hat{F}}}{\hat{N}!}\int_0^{2\pi}\frac{d\lambda_1}{2\pi}\cdots\int_0^{2\pi}\frac{d\lambda_{\hat{N}}}{2\pi}\, f(\lambda_1,\ldots,\lambda_{\hat{N}}) \times \prod_{i<j} 4\sin^2\frac{\lambda_i-\lambda_j}{2} \times \sum_{L=-\infty}^{+\infty} e^{iL(\lambda_1+\cdots+\lambda_{\hat{N}})} \tag{3.12}$$

where the last factor is the δ-function for the $\sum_i \lambda_i$ modulo 2π.

Thus far we made only exact calculation, but now we turn to approximations valid in the $\hat{N}\to\infty$ limit. Re-writing the integrand of eq. (3.12) in exponential form

$$\exp\left[\begin{array}{l} -\dfrac{\hat{N}}{\hat{S}}\displaystyle\sum_{j=1}^{\hat{N}}\mathcal{W}(e^{i\lambda_j}) - \sum_{j=1}^{\hat{N}}\sum_{f=1}^{\hat{F}}\log\left(e^{i\lambda_j}-\left(\frac{\hat{\mu}_f}{\hat{v}}\right)^K\right)+ \\ \qquad\qquad\qquad + \displaystyle\sum_{i<j}\log\left(4\sin^2\frac{\lambda_i-\lambda_j}{2}\right) + iL\sum_{j=1}^{\hat{N}}\lambda_j \end{array}\right], \tag{3.13}$$

we see that all terms in exponent grow with $\hat{N}$. Hence, in the large $\hat{N}$ limit we may use the saddle-point approximation:

$$\begin{aligned}\log\mathcal{Z} \approx -\frac{\hat{N}}{\hat{S}}\sum_{j=1}^{\hat{N}}\mathcal{W}(e^{i\bar{\lambda}_j}) - \sum_{j=1}^{\hat{N}}\sum_{f=1}^{\hat{F}}\log\left(e^{i\bar{\lambda}_j}-\left(\frac{\hat{\mu}_f}{\hat{v}}\right)^K\right) + \\ + \sum_{i<j}\log\left(4\sin^2\frac{\bar{\lambda}_i-\bar{\lambda}_j}{2}\right) + i\overline{L}\sum_{j=1}^{\hat{N}}\bar{\lambda}_j\end{aligned} \tag{3.14}$$

where $(\bar{\lambda}_1,\ldots,\bar{\lambda}_{\hat{N}})$ and $\overline{L}$ maximize the right hand side of this formula, or rather maximize its real part and extremize it imaginary part; generally, this requires moving away from the real axis into the complex plane. Or into some other complex space: since the λ_i are periodic variables on the circle $\mathbb{R}/2\pi\mathbb{Z}$, the $\bar{\lambda}_i$ move into the complex cylinder $\mathbb{C}/2\pi\mathbb{Z}$ rather than into the plane $\mathbb{C}$. As to the L, in the large $\hat{N}$ limit the maximum happens for $L=O(\hat{N})$ where the discreteness of L does not matter any more, hence $\overline{L}=\frac{\hat{N}}{\hat{S}}\times\hat{C}$ where $\hat{C}$ is finite and complex.

As usual for matrix models, for $\hat{N}\to\infty$ the spectrum Σ of $(\bar{\lambda}_1,\ldots,\bar{\lambda}_{\hat{N}})$ becomes continuous. In general, it comprises several continuous line segments $\Sigma_1,\ldots,\Sigma_n$ on the complex cylinder. Denoting the spectral density over those lines by $\frac{\hat{N}}{\hat{S}}\rho(\lambda)d\lambda$, we write the free energy of the matrix model as

$$\mathcal{F} \equiv -\frac{\hat{S}^2}{\hat{N}^2}\log\mathcal{Z} = \mathcal{F}_S + \frac{\hat{S}}{\hat{N}}\mathcal{F}_D + O\left(\frac{1}{\hat{N}^2}\right), \tag{3.15}$$

where

$$\mathcal{F}_S = \int_{\substack{\text{whole}\\ \Sigma}} d\lambda\,\rho(\lambda)\,\left[\mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda\right] - \frac{1}{2}\int_{\substack{\text{whole}\\ \Sigma}} d\lambda\,\rho(\lambda)\int_{\substack{\text{whole}\\ \Sigma}} d\lambda'\,\rho(\lambda')\,\log\left(4\sin^2\frac{\lambda-\lambda'}{2}\right) \tag{3.16}$$

and

$$\mathcal{F}_D = \sum_{f=1}^{\hat{F}}\int_{\substack{\text{whole}\\ \Sigma}} d\lambda\,\rho(\lambda)\,\log\left(e^{i\lambda} - \left(\frac{\hat{\mu}_f}{\hat{v}}\right)^K\right). \tag{3.17}$$

In "worldsheet" terms of the $1/\hat{N}$ expansion [14], the leading contribution $\mathcal{F}_S$ which comes from $\mathcal{W}_{\text{def}}$ and the Vandermonde determinant corresponds to genus $g = 0$ *i.e.* spherical topology, hence the notation. Likewise, the quarks' sector contribution corresponds to the disk topology, hence the notation $\mathcal{F}_D$.

The spectral density $\rho(\lambda)$ minimizes the free energy $\mathcal{F} \approx \mathcal{F}_S$ under constraint

$$\#(\bar{\lambda}_i) = \hat{N} \Longrightarrow \int_{\substack{\text{whole}\\ \Sigma}} d\lambda\,\rho(\lambda) = \hat{S}\,, \tag{3.18}$$

and this gives us a variational equation

$$\frac{d}{d\lambda}\,\frac{\delta\mathcal{F}_S}{\delta\rho(\lambda)} = 0\,. \tag{3.19}$$

In light of eq. (3.16), this equation becomes

$$\frac{d\mathcal{W}}{d\lambda} - i\hat{C} - \fint_{\substack{\text{whole}\\ \Sigma}} d\lambda'\,\rho(\lambda')\,\cot\frac{\lambda-\lambda'}{2} = 0 \tag{3.20}$$

where the bar across the integral sign indicates principal-value integration over the pole at $\lambda' = \lambda$. However, it is more convenient to regulate the pole via bypassing it in the complex plane, so let us introduce the resolvent

$$\hat{R}(w) = \int_{\substack{\text{whole}\\ \Sigma}} d\lambda\,\rho(\lambda)\,\cot\frac{w-\lambda}{2} \tag{3.21}$$

which has branch cuts along the spectral segments $\Sigma_1, \ldots, \Sigma_n$ but is analytic elsewhere in the complex cylinder. In terms of this resolvent, eq. (3.20) applies to the average of the two sides of any branch cut, hence

$$\forall\lambda\in\Sigma: \quad \hat{R}(\lambda \pm i\epsilon) = \frac{d\mathcal{W}}{d\lambda} - i\hat{C} \mp 2\pi i\rho(\lambda)\,. \tag{3.22}$$

Together, eqs. (3.21)–(3.22) lead to a quadratic equation for the resolvent $\hat{R}(w)$:

$$\hat{R}^2(w) - 2\left[\frac{d\mathcal{W}(e^{iw})}{dw} - i\hat{C}\right]\times\hat{R}(w) + 2F(e^{iw}) + \hat{S}^2 = 0 \tag{3.23}$$

where F is a polynomial function of e^{iw}.[5] Physically, eq. (3.23) is nothing but the loop equation (2.31) written in terms of the periodic coordinate w of the complex cylinder instead of the flat coordinate $X = e^{iw}$. Indeed, in the saddle point approximation we have

$$\langle R(X)\rangle = \frac{\hat{S}}{\hat{N}} \sum_i \frac{1}{X - \exp(i\bar{\lambda}_i)} = \int_{\substack{\text{whole}\\ \Sigma}} d\lambda \, \frac{\rho(\lambda)}{X - e^{i\lambda}} \tag{3.25}$$

and hence

$$\hat{R}(w) = 2iX\,\langle R(X)\rangle - i\hat{S} \qquad \text{for } X = e^{iw}\,. \tag{3.26}$$

Likewise

$$F(X = e^{iw}) = \hat{S} \times \big(\langle C\rangle - X\mathcal{W}'(X) + 2X\,\langle P(X)\rangle\big)\,, \tag{3.27}$$

and if we also identify $\hat{C} = \langle C\rangle$ then eq. (3.23) becomes the loop equation (2.31).

Solving the loop equations gives us

$$\langle R(X)\rangle = \frac{X\mathcal{W}'(X) + \hat{S} - \hat{C} - Y}{2X} \Longrightarrow \hat{R} = iX\mathcal{W}'(X) - i\hat{C} - iY(X) \tag{3.28}$$

where

$$Y(X) = \pm\sqrt{\Big(X\mathcal{W}'(X) + \hat{S} - \hat{C})^2 - 4\hat{S}X\Big(\mathcal{W}'(X) - \langle P(X)\rangle\Big)} \tag{3.29}$$

has $n \le d = \text{degree}(\mathcal{W})$ branch cuts which connect simple zeroes of the discriminant $Y^2(X)$. Let us denote such zeroes a_i^- and a_i^+ $(i = 1, \ldots, n)$[6] and map them into the w cylinder as $a_i^{\pm} = \exp(i\theta_i^{\pm})$. Consequently, the $\bar{\lambda}$ spectrum Σ has precisely n segments Σ_i which begin at θ_i^- and end at θ_i^+, the spectral density along each segment is given by

$$\rho(\lambda) = \frac{1}{2\pi}\, Y(X = e^{i\lambda}) \tag{3.30}$$

(cf. eqs. (3.22) and (3.28)), and the shapes of the segments follow from $\rho(\lambda)d\lambda$ being real and positive.

From the field theory point of view, the segments Σ_i correspond to confining (or rather pseudo-confining) subgroups $\mathrm{SU}(N_i)$ of the diagonal $\mathrm{SU}(N_c)$, and the integrals

$$\hat{S}_i = \int_{\theta_i^-}^{\theta_i^+} d\lambda\, \rho(\lambda) = \frac{1}{4\pi i} \oint\limits_{A_i} dw\, \hat{R}(w) = \frac{1}{2\pi i} \oint\limits_{A_i} dX\, \langle R(X)\rangle \tag{3.31}$$

[5]Integrating $\int d\lambda' \rho(\lambda') \int d\lambda'' \rho(\lambda'')$ the trigonometric identity

$$\cot\frac{\lambda \pm i\epsilon - \lambda'}{2} \times \cot\frac{\lambda' \pm i\epsilon - \lambda''}{2} - 1 = \cot\frac{\lambda \pm i\epsilon - \lambda''}{2} \times \left(\cot\frac{\lambda \pm i\epsilon - \lambda'}{2} + \cot\frac{\lambda' \pm i\epsilon - \lambda''}{2}\right),$$

we arrive after some work at eq. (3.23) where

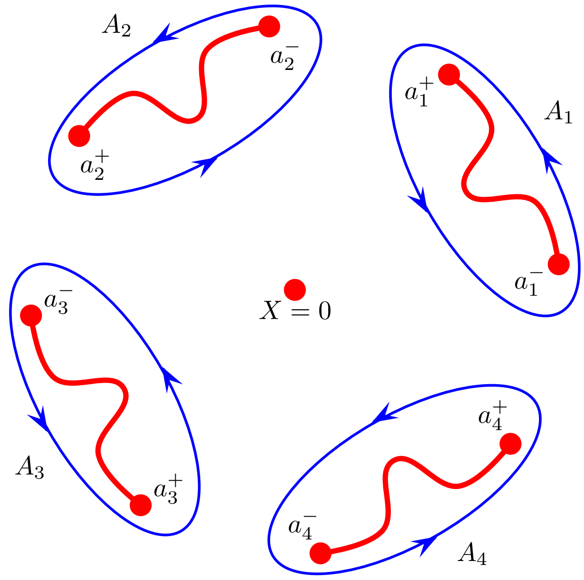
$$F(w) = \int_\Sigma d\lambda\, \rho(\lambda)\, \cot\frac{w - \lambda}{2} \times \left(\frac{d\mathcal{W}(e^{iw})}{dw} - \frac{d\mathcal{W}(e^{i\lambda})}{d\lambda}\right). \tag{3.24}$$

[6]From now on we use index i to label spectral segments Σ_i and related parameters rather than individual eigenvalues.

are dual to the gaugino condensates

$$S_i = \operatorname{Tr}\left(\frac{W^\alpha W_\alpha}{32\pi^2}\right)\Bigg|_{\mathrm{SU}(N_i)} = \frac{1}{2\pi i}\oint\limits_{A_i} dX\, R(X)\,. \tag{3.32}$$

In both cases the A_i are A-cycles of the Riemann surface of the $R(X)^{\text{field}}_{\text{theory}} \leftrightarrow \langle R(X)\rangle^{\text{matrix}}_{\text{model}}$ or their images on the w cylinder; the A_i surround individual segments of the spectrum as shown below:



(3.33)

The advantage of the contour integral formalism is that we do not need to know the exact routes of all the segments but only the general locations of the A-cycles which surround them. This works for all integrals of the form $\int d\lambda\rho(\lambda)f(\lambda)$, for all kinds of $f(\lambda)$ functions. For example, eq. (3.17) for the disk-level free energy can be written as a sum of contour integrals

$$\mathcal{F}_D = \sum_{f=1}^{\hat F}\frac{1}{4\pi i}\sum_{i=1}^{n}\oint\limits_{A_i} dw\, \hat R(w)\times\log\left(e^{iw}-\left(\frac{\hat\mu_f}{\hat v}\right)^K\right). \tag{3.34}$$

The sphere-level free energy can also be written as a sum of contour integrals, but this is more complicated because the logarithm in the second terms in eq. (3.16) has branch cuts of its own. To disentangle the branch cut structure, let us first rewrite eq. (3.16) as

$$\mathcal{F}_S = \frac{1}{2}\int_{\substack{\text{whole}\\ \Sigma}} d\lambda\,\rho(\lambda)\times\left[\mathcal{W}(e^{i\lambda}) - i\hat C\lambda + \hat E(\lambda)\right] \tag{3.35}$$

where

$$\hat E(\lambda) = \mathcal{W}(e^{i\lambda}) - i\hat C\lambda - \int_{\substack{\text{whole}\\ \Sigma}} d\lambda'\,\rho(\lambda')\times\log\left(4\sin^2\frac{\lambda-\lambda'}{2}\right). \tag{3.36}$$

Naively,

$$\frac{\delta \mathcal{F}_S}{\delta \rho(\lambda)} = \hat{E}(\lambda) \Longrightarrow \frac{d\hat{E}}{d\lambda} = 0 \tag{3.37}$$

(cf. eq. (3.19)), but analytic continuation to the complex cylinder yields instead

$$\frac{d\hat{E}}{d\lambda} = \frac{d\mathcal{W}}{d\lambda} - i\hat{C} - \hat{R}(\lambda) = iY(X = e^{i\lambda}) \neq 0 \tag{3.38}$$

and hence the $\hat{E}(\lambda)$ itself is a non-trivial function with branch cuts along the Σ_i segments. However, its *average* between the two sides of a cut is locally constant

$$\frac{\hat{E}(\lambda + i\epsilon) + \hat{E}(\lambda - i\epsilon)}{2} \equiv \text{constant } \hat{E}_i \qquad \text{for } \lambda \in \Sigma_i \text{ only} \tag{3.39}$$

because the derivative (3.38) flips sign across the cut. Therefore, to make sure eq. (3.19) is consistent with the loop equation, we must disambiguate the logarithm in the second term of eq. (3.16) such that

$$\text{for } \lambda \in \Sigma\,, \qquad \frac{\delta \mathcal{F}_S}{\delta \rho(\lambda)} = \frac{\hat{E}(\lambda + i\epsilon) + \hat{E}(\lambda - i\epsilon)}{2}\,. \tag{3.40}$$

Hence in light of eq. (3.39),

$$\begin{aligned} \mathcal{F}_S &= \frac{1}{2}\sum_{i=1}^{n}\int_{\Sigma_i} d\lambda\, \rho(\lambda) \times \left[\mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda + \hat{E}_i\right] \\ &= \frac{1}{2}\sum_{i=1}^{n}\left[\hat{E}_i \times \hat{S}_i + \frac{1}{4\pi i}\oint_{A_i} dw\, \hat{R}(w) \times \left(\mathcal{W}(e^{iw}) - i\hat{C}w\right)\right]. \end{aligned} \tag{3.41}$$

The $\hat{E}_i$ can also be calculated as contour integrals. For general λ,

$$\hat{E}(\lambda) = \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{4\pi i}\sum_{i=1}^{n}\oint_{A_i} dw\, \hat{R}(w) \times \log\left(4\sin^2\frac{\lambda - w}{2}\right) \tag{3.42}$$

where the contours A_i should be drawn such as to exclude the branch cuts of the logarithm. In particular, the λ point itself should be kept outside of all the A_i contours, and this prevents us from directly evaluating eq. (3.42) for $\lambda \in \Sigma$. Instead, for the purpose of calculating an $\hat{E}_i$ in eq. (3.41) we must first evaluate the integral eq. (3.42) for $\lambda \notin \Sigma$, then take two limits of λ approaching the same point $\lambda_i \in \Sigma_i$ from two opposite sides of the spectrum, and finally take the average of the two limits. Alternatively, we may take just one limit of λ going to an end point θ_i^+ or θ_i^- of the spectral segment — at these points the difference between the two sides of the spectrum vanishes and the averaging becomes unnecessary. Thus,

$$\hat{E}_i = \lim_{\lambda \to \theta_i^{\pm}}\left[\mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{4\pi i}\sum_{j=1}^{n}\oint_{A_j} dw\, \hat{R}(w) \times \log\left(4\sin^2\frac{\lambda - w}{2}\right)\right]. \tag{3.43}$$

In the next section, these equations (as well as eq. (3.34)) will help us calculate the effective superpotential of the matrix model.

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We conclude this section with a brief discussion of local *versus* global minima of the free energy. Eqs. (3.41) and (3.43) give us the $\mathcal{F}_S$ minimized with respect to *local* variations of the spectral density $\rho(\lambda)$, hence eq. (3.30). The global minimum requires further minimization with respect to the free parameters of eq. (3.29), namely $\hat{C}$ and $n-1$ independent coefficients of the $\langle P(X)\rangle$ polynomial.[7] Variationally, this implies that $\delta\mathcal{F}_S/\delta\rho(\lambda)$ should be globally constant over the whole spectrum Σ and not just individual segments, hence

$$\hat{E}_1 = \hat{E}_2 = \cdots = \hat{E}_n\,. \tag{3.44}$$

In addition, we should minimize with respect to the $\hat{C}$ parameter, hence

$$\frac{\delta\mathcal{F}_S}{\delta\hat{C}} = \int_{\substack{\text{whole}\\ \Sigma}} d\lambda\,\rho(\lambda)\times(-i\lambda) = -\frac{1}{4\pi}\sum_{i=1}^{n}\oint_{A_i} dw\,\hat{R}(w)\times w = 0\,. \tag{3.45}$$

Note that eqs. (3.44) apply only to the global minimum of the free energy; in field theory terms this corresponds to taking the gaugino condensates S_i on-shell. In the following section, we shall calculate the W_{eff} for the off-shell S_i or rather $\hat{S}_i$, and this means abandoning the global minimum and hence eqs. (3.44). Instead, eqs. (3.31) will determine the coefficients of $\langle P(X)\rangle$ in terms of the $\hat{S}_i$.

4. The effective superpotential

In this section we derive the effective superpotential of our matrix model. But first, a few general words about effective superpotentials for the off-shell gaugino condensates in field theory. In the single-U(N) theory with adjoint matter, Cachazo et al. [6] found that the gauginos of the U(1) center of the U(N) generate auxiliary supersymmetries of the chiral ring of the gauge theory and hence

$$W_{\text{eff}}(S_1,\ldots,S_n) = \sum_{i=1}^{n} N_i\frac{\partial}{\partial S_i}\mathcal{F}(S_1,\ldots,S_n) + \sum_{i=1}^{n}(2\pi i\tau_0 + b_i)\times S_i \tag{4.1}$$

where $\mathcal{F}(S_1,\ldots,S_n)$ is a prepotential of the auxiliary SUSY, τ_0 is the overall bare gauge coupling, and b_i are integers distinguishing between specific vacua of the theory.[8] Under gauge-matrix duality, τ_0 is just an arbitrary parameter, but the prepotential $\mathcal{F}$ is dual to the sphere-level free energy $\mathcal{F}_S(\hat{S}_1,\ldots,\hat{S}_n)$ of the matrix model.

Adding the quarks — which couple to the U(1) center — breaks the auxiliary supersymmetries, but the subring of the adjoint sector remains supersymmetric. Hence [14], integrating out the mesonic sector of the chiral ring yields

$$W_{\text{eff}}(S_1,\ldots,S_n) = \sum_{i=1}^{n} N_i\frac{\partial}{\partial S_i}\mathcal{F}(S_1,\ldots,S_n) + \sum_{i=1}^{n}(2\pi i\tau_0 + b_i)\times S_i + \mathcal{F}_Q(S_1,\ldots,S_n) \tag{4.2}$$

where $\mathcal{F}_Q$ is the quark sector's contribution to the superpotential. Under gauge-matrix duality, the $\mathcal{F}_Q$ is dual to the disk-level part $\mathcal{F}_D$ of the matrix model' free energy.

[7] $\langle P(X)\rangle$ has degree $d-2$ and hence $d-1$ coefficients, but if we want $n<d$ spectral segments, the polynomial $Y^2(X)$ (cf. eq. (3.29)) must have $d-n$ double zeroes, which imposes $d-n$ constraints on the coefficients of $\langle P(X)\rangle$. Consequently, only $n-1$ of those coefficients may vary independently of each other.

[8] By analogy, in the Veneziano-Yankielowicz superpotential $W = NS\log S + (2\pi i\tau_0 + b)S$, b distinguishes between different vacua of the SU(N).

In our case, the $[\mathrm{SU}(N)]^K$ theory does not have a U(1) center(s), and we cannot promote the gauge symmetry to $[\mathrm{U}(N)]^K$ because of gauge anomaly constraints, but we can add a single U(1) factor common to all nodes of the quiver diagram (1.3). Since all the gaugino condensates of our theory belong to subgroups of the diagonal SU(N), adding the common U(1) effectively promotes the $\mathrm{SU}(N)_{\mathrm{diag}}$ to the $\mathrm{U}(N)_{\mathrm{diag}}$, with similar consequences for the effective superpotential. Thus, $W_{\mathrm{eff}}(S_1,\ldots,S_n)$ is given by eq. (4.2) where the $\mathcal{F}(S_1,\ldots,S_n)$ is dual to the $\mathcal{F}_S$ of our matrix model and the $\mathcal{F}_Q(S_1,\ldots,S_n)$ is dual to its $\mathcal{F}_D$.

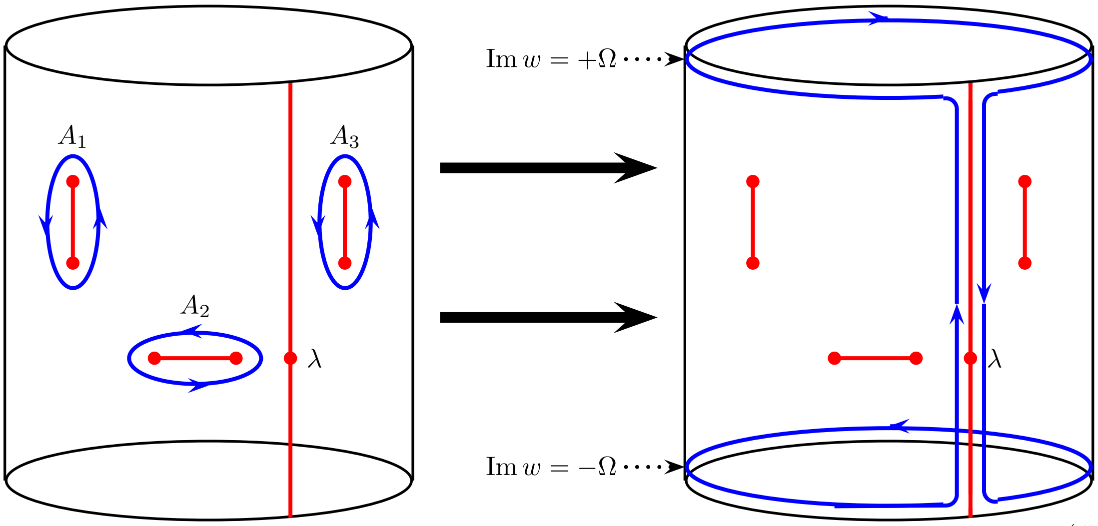
In the matrix model, taking the $\hat{S}_i$ off-shell and evaluating the free energy as a function $\mathcal{F}(\hat{S}_1,\ldots,\hat{S}_n)$ means imposing all n eqs. (3.31) as constraints and then minimizing the $\mathcal{F}\approx\mathcal{F}_S$ under those constraints. Variationally, this implies that $\delta\mathcal{F}_S/\delta\rho(\lambda)$ should be locally constant along each continuous segment Σ_i of the spectrum but may take different values for different segments. Therefore, eq. (3.30) and all the subsequent contour-integral formulæ of section 3 remain unchanged, but we throw eqs. (3.44) out of the window and instead use eqs. (3.31) to determine the coefficients of $\langle P(X)\rangle$. Consequently, exactly as in [6]

$$\frac{\partial\mathcal{F}_S}{\partial\hat{S}_i}=\frac{\delta\mathcal{F}_S}{\delta\rho(\lambda)}\bigg|_{\lambda\in\Sigma_i}=\hat{E}_i\,, \tag{4.3}$$

and hence the effective superpotential of the matrix model is given by

$$W_{\mathrm{eff}}(\hat{S}_1,\ldots,\hat{S}_n)=\sum_{i=1}^{n}\hat{N}_i\times\hat{E}_i(\hat{S}_1,\ldots,\hat{S}_n)+\sum_{i=1}^{n}(2\pi i\tau_0+b_i)\times\hat{S}_i+\mathcal{F}_D(\hat{S}_1,\ldots,\hat{S}_n). \tag{4.4}$$

In the contour integral formalism, the $\hat{E}_i$ in this formula are given by eqs. (3.43). We can simplify the contour integrals by merging all the A_i cycles together and then pushing the resulting loop outwards. The following picture illustrates how this works on the complex cylinder:



(4.5)

The new contour comprises two vertical lines on two sides of the logarithm's branch cut, and two loops around the cylinder. For the loops, we take $\Omega \to \infty$ which gives us

$$\hat{R}(w) \xrightarrow[\operatorname{Im}(w)\to\pm\infty]{} \mp i\hat{S} \tag{4.6}$$

and therefore

$$\begin{aligned} \text{top loop} &= 2\pi i\hat{S}\left(\Omega - \operatorname{Im}(\lambda_i)\right), & (4.7)\\ \text{bottom loop} &= 2\pi i\hat{S}\left(\Omega + \operatorname{Im}(\lambda_i)\right). & (4.8)\end{aligned}$$

For the vertical lines, we have discontinuity

$$\operatorname{disc}\left[\log\left(4\sin^2\frac{\lambda - w}{2}\right)\right] = \begin{cases} +2\pi i & \text{for } \operatorname{Im} w > \operatorname{Im}\lambda, \\ -2\pi i & \text{for } \operatorname{Im} w < \operatorname{Im}\lambda, \end{cases} \tag{4.9}$$

and hence

$$\int_{\substack{\text{vert.}\\ \text{lines}}} dw\,\hat{R}(w) \times \log(\cdots) = 2\pi i \int_\lambda^{+i\Omega} dw\,\hat{R}(w) + 2\pi i \int_\lambda^{-i\Omega} dw\,\hat{R}(w)\,. \tag{4.10}$$

Putting the whole contour integral together, we arrive at

$$\begin{aligned} \hat{E}(\lambda) &= \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{2}\int_\lambda^{+i\Omega} dw\,\hat{R}(w) - \frac{1}{2}\int_\lambda^{-i\Omega} dw\,\hat{R}(w) - \hat{S}\times\Omega \\ &= \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{2}\int_\lambda^{+i\infty} dw\,[\hat{R}(w) + i\hat{S}] - \frac{1}{2}\int_\lambda^{-i\infty} dw\,[\hat{R}(w) - i\hat{S}]\,, \end{aligned} \tag{4.11}$$

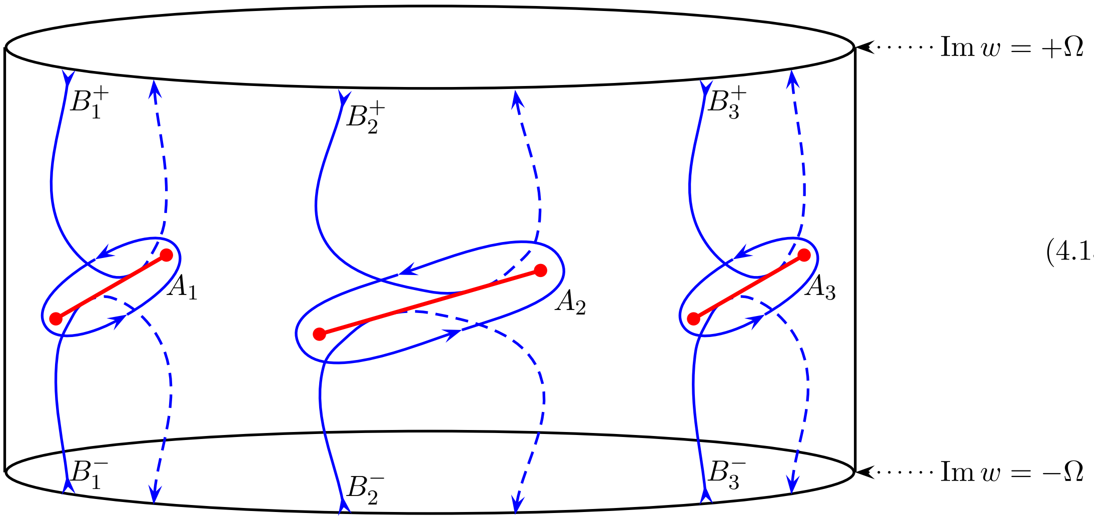
and at this point, we may take $\lambda = \theta_i^+$ or $\lambda = \theta_i^-$ to calculate the $\hat{E}_i$.

We may also express the $\hat{E}_i$ in term of the B-cycle periods of the Riemann surface of $Y(X = e^{iw})$. Indeed, plugging eq. (3.28) into formulæ (4.11), we obtain

$$\begin{aligned} \hat{E}_i &= \frac{i}{2}\int_{\theta_i^\pm}^{+i\Omega} dw\,Y(e^{iw}) + \frac{i}{2}\int_{\theta_i^\pm}^{-i\Omega} dw\,Y(e^{iw}) + \frac{1}{2}\mathcal{W}(e^{iw}) - \hat{S}\Omega \\ &= -\frac{i}{4}\int_{B_i^+}^{\text{reg}} dw\,Y(e^{iw}) - \frac{i}{4}\int_{B_i^-}^{\text{reg}} dw\,Y(e^{iw}) + \frac{1}{2}\mathcal{W}(e^{iw}) - \hat{S}\Omega\,, \end{aligned} \tag{4.12}$$

where the $B_i^\pm$ cycle begins at $w = \pm i\infty$ on the physical sheet of the Riemann surface, crosses the Σ_i branch cut to the other sheet, and then goes back to $w = \pm i\infty$ but on the unphysical sheet, and the integrals are regularized by starting and stopping at $w = \pm i\Omega$

instead of $w = \pm i\infty$. The cycles are illustrated on the following figure:



(4.13)

Note that the each sheet of our Riemann surface is a cylinder with two distinct infinities — and that's why we have a double set of B-cycles. Fortunately, most of the extra B cycles are redundant:

$$\forall i,j: \quad B_i^+ - B_i^- \equiv B_j^+ - B_j^- \quad \text{modulo } A \text{ cycles}. \tag{4.14}$$

and hence

$$\begin{aligned}\hat{E}_i = -\frac{i}{2}\int_{B_i^-}^{\text{reg}} dw\, Y(e^{iw}) + \text{terms common to all } \hat{E}_j + \\ + \text{a linear combination of } \hat{S}_j \text{ with integer coefficients}.\end{aligned} \tag{4.15}$$

The last term here reflects the ambiguity of B-cycles modulo A cycles: going from a branch cut Σ_i to $\pm i\infty$ one may choose different passages between the other branch cuts Σ_j. In terms of the sphere-level free energy, this corresponds to different routing of the branch cuts of the log $\left(4\sin^2\frac{\lambda-\lambda'}{2}\right)$ around the spectral segments Σ_j in both λ and λ' complex cylinders. Re-routing the log's branch cuts changes the free energy by

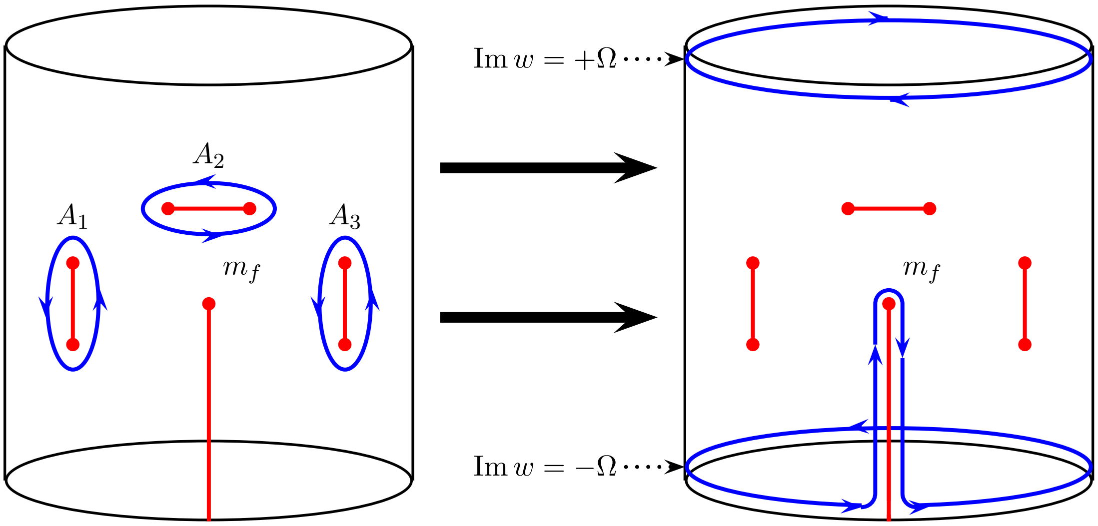
$$\mathcal{F}_S(\hat{S}_1,\ldots,\hat{S}_n) \to \mathcal{F}_S(\hat{S}_1,\ldots,\hat{S}_n) + \frac{1}{2}\sum_{i,j} c_{ij}\hat{S}_i\hat{S}_j \tag{4.16}$$

for some *integer* coefficients c_{ij}, and hence

$$\hat{E}_i(\hat{S}_1,\ldots,\hat{S}_n) \to \hat{E}_i(\hat{S}_1,\ldots,\hat{S}_n) + \sum_j c_{ij}\hat{S}_j\,. \tag{4.17}$$

Fortunately, from the superpotential (4.4) point of view, this change amounts to changing the integers b_i. In other words, we permute the branches belonging to different vacua, but the overall picture does not change.

Now consider the quark sector's contribution $\mathcal{F}_D(\hat{S}_1,\ldots,\hat{S}_n)$ to the effective superpotential. Again, we are going to simplify eq. (3.34) by moving the integration contours, but it's convenient to do it flavor-by-flavor. We need to distinguish between massive and massless flavors: they play different roles in field theory, both in 4D and in 5D,[9] and in the matrix model they have different contributions to the disk-level free energy. Let us begin with a massive flavor and let $e^{im_f} = (\hat{\mu}_f/\hat{v})^K$. Then for this flavor we have cuts and contours as shown on the left picture below:



(4.18)

Again, we merge the A_i contours into a single loop and push it outward as shown on the right picture above. This time, we end up with two disconnected contours: a simple loop around the top of the cylinder, and a loop around the cylinder's bottom attached to a vertical loop around the logarithm's cut. The discontinuity across this cut is simply $+2\pi i$ and hence

$$\oint_{\substack{\text{vert.}\\ \text{loop}}} dw\, \hat{R}(w) \times \log\left(e^{iw} - e^{im_f}\right) = 2\pi i \int_{-i\Omega}^{m_f} dw\, \hat{R}(w). \tag{4.19}$$

And for the loops around the cylinder we again take $\Omega \to \infty$ and use eq. (4.6); consequently

$$\text{top loop} = 2\pi i \hat{S} \times (im_f - \pi i)\,, \qquad \text{bottom loop} = 2\pi i \hat{S} \times (\Omega + \pi i)\,. \tag{4.20}$$

Putting all the loops together, we find that one massive quark flavor contributes

$$\mathcal{F}_D|^{\text{one}\,\mu\neq 0}_{\text{flavor}} = \frac{1}{2}\int_{-i\Omega}^{m_f} dw\, \hat{R}(w) + \frac{\hat{S}}{2} \times (\Omega + im_f) = \frac{1}{2}\int_{-i\infty}^{m_f} dw\, [\hat{R}(w) + i\hat{S}] + i\hat{S} \times m_f\,. \tag{4.21}$$

For a massless flavor we have $\log(e^{iw} - (0/\hat{v})^K) = iw$, which does not have singularities on the complex cylinder but needs a branch cut anyway because w is multi-valued.

[9] In deconstruction [11], massive 4D flavors with $\mu_f \approx \hat{v}$ have light modes and give rise to light flavors in 5D; the massless 4D flavors do not have no 5D counterparts, but they are needed to adjust the Chern-Simons level of the 5D theory. From the purely 4D point of view [12], the massless flavors affect the Coulomb branch of the theory but do not give rise to Higgs branches; the on-shell mesonic and baryonic operators include only the massive flavors.

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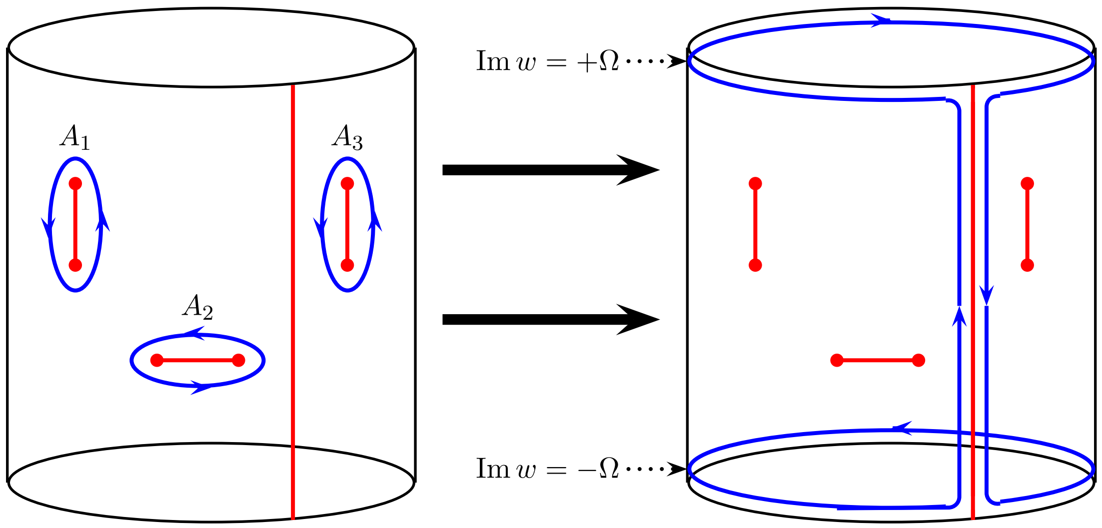
Consequently, we have contours and cuts as shown below:



$$(4.22)$$

Once again we merge the contours into a single loop and push it outward as shown on the right picture above. This time we get a connected system of two loops around the cylinder and two vertical lines, and evaluating the integrals gives us

$$\text{vertical lines}: \Longrightarrow 2\pi i \int_{-i\Omega}^{+i\Omega} dw\, \hat{R}(w)\,, \tag{4.23}$$

$$\text{each horizontal loop}: \Longrightarrow 2\pi i S \times (\Omega \pm \pi i)\,, \tag{4.24}$$

hence altogether one massless flavor yields

$$\mathcal{F}_D|_{\text{flavor}}^{\text{one}\,\mu=0} = \frac{1}{2}\int_{-i\infty}^{0} dw\, [\hat{R}(w) + i\hat{S}] + \frac{1}{2}\int_{0}^{+i\infty} dw\, [\hat{R}(w) - i\hat{S}]\,. \tag{4.25}$$

We conclude this section with a complete formula for the effective superpotential. Combining eqs. (4.4), (4.11), (4.21), and (4.25) together, we arrive at

$$\begin{aligned} W_{\text{eff}} = & \sum_{i=1}^{n} \hat{N}_i \times \left(\mathcal{W}(e^{i\theta_i^+}) - i\hat{C}\theta_i^+ + \int_{-i\Omega}^{i\theta_i^+} dw\, \hat{R}(w) \right) + \\ & + \frac{1}{2}\sum_{f=1}^{\hat{F}_1} \int_{-i\Omega}^{m_f} dw\, \hat{R}(w) - \frac{\hat{N} - \hat{F}_2}{2} \times \int_{-i\Omega}^{+i\Omega} dw\, \hat{R}(w) + \\ & + 2\pi i\tau \times \hat{S}_{\text{net}} + \sum_{i=1}^{n} b_s \hat{S}_i \end{aligned} \tag{4.26}$$

where $\hat{F}_1$ is the number of massive flavors, $\hat{F}_2$ is the number of massless flavors, and τ is the renormalized gauge coupling according to

$$2\pi i\tau = 2\pi i\tau_0 + \frac{1}{2}\sum_{f=1}^{\hat{F}_1} \log\frac{\hat{\mu}_f^K}{\hat{v}^K} - \Omega \times \left(\hat{N} - \frac{1}{2}\hat{F}_1 - \hat{F}_2 \right). \tag{4.27}$$

Note the coefficient of the cutoff Ω in this renormalization: The same combination $N - \frac{1}{2}F_1 - F_2$ appears in dimensional deconstruction as the coefficient of the 5D Chern-Simons term. We are not sure of the physical meaning of this coincidence, but it probably isn't an accident.

Eq. (4.26) gives the effective superpotential in terms of the resolvent $\hat{R}(w)$, and to recast it as a function of the gaugino condensates $\hat{S}_1, \ldots, \hat{S}_n$ we need to solve eqs. (3.31) for the resolvent's parameters. Since the resolvent with n branch cuts has $n+1$ independent parameters (including the $\hat{S} = \sum_i \hat{S}_i$), determining all of them requires one more equation besides n eqs. (3.31). In the matrix model, such $n+1^{\underline{\rm st}}$ equation is eq. (3.45), which follows from minimizing the free energy with respect to $\hat{C}$. The contour integrals in eq. (3.45) are similar to a massless flavor's contribution to the $\mathcal{F}_D$ and can be simplified in the same way: moving the contours according to figure (4.22), we arrive at

$$\frac{\partial \mathcal{F}_S}{\partial \hat{C}} = -\frac{1}{2} \int_{-i\Omega}^{+i\Omega} dw\, \hat{R}(w) - \Omega \hat{S} = 0\,. \tag{4.28}$$

The the effective superpotential $W^{\rm eff}(\hat{S}_1, \ldots, \hat{S}_n)$ as a function of the gaugino condensates follows from *combined* eqs. (4.28), (3.31), and (4.26).

5. Open questions

We conclude this paper by discussing open questions raised by our unitary matrix model.

The first question is specific to the $[\mathrm{SU}(N)]^K$ model: what, if anything, is the gauge-theory dual of eq. (4.28)? The chiral ring of the quiver (1.3) was studied in detail in [12], and none of the anomaly equations there looks remotely like eq. (3.45) or eq. (4.28). Instead, the $n + 1^{\underline{\rm st}}$ equation for the parameters of the $R(X)$ resolvent comes from a completely different source, namely the loop equations for the link resolvent $T(X) = \mathrm{tr}\left(\frac{1}{X - \Omega_K \cdots \Omega_1}\right)$. The analytic properties of the two resolvents $R(X)$ and $T(X)$ require $Y(X)$ (cf. eq. (3.29)) to have the same branching points in the X plane as

$$\sqrt{\prod_{j=1}^{N_c} (X - \varpi_j)^2 - 4\left((-\gamma)^{N_f} \Lambda^{2N_c - N_f}\right)^K \times \prod_{f=1}^{N_f} (\mu_f^K - X)} \tag{5.1}$$

where $\varpi_1 \ldots, \varpi_{N_c}$ are the Coulomb moduli of the $[\mathrm{SU}(N)]^K$ theory; only $N_c - 1$ of these moduli are independent because of the $\mathbf{S}\mathrm{U}(N_c)$ constraint

$$\prod_{j=1}^{N_c} \varpi_j = \hat{v}^{KN_c}\,. \tag{5.2}$$

Together, eqs. (5.1)–(5.2) impose one constraint on the parameters of the gaugino-condensate resolvent $R(X)$; the remaining parameters are related to the individual condensates S_i according to eqs. (3.32), exactly as in the matrix model.

Unfortunately, the link resolvent $T(X)$ and the Coulomb moduli $\varpi_1 \ldots, \varpi_{N_c}$ of the gauge theory do not have any matrix duals. Consequently, eqs. (5.1)–(5.2) do not make

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sense on the matrix-model side of the duality, just like eq. (4.28) does not make any sense on the gauge-theory side. Ideally, these equations could be dual to each other, but we do not have any evidence for such duality. In fact, we do not even know whether eqs. (5.1)–(5.2) and (4.28) are even mutually consistent. This remains an open question we hope to answer as soon as possible.

A bigger open question concerns generalization of our unitary matrix model to other quiver theories. We believe that the random unitary matrices can be used to model all kinds of chiral $(\mathbf{n}, \bar{\mathbf{n}})$ bifundamental fields with non-zero eigenvalues, but we would like to see how this works in different models. Also, it would be interesting to see what exactly goes wrong when an eigenvalue happens to vanish in some vacuum state of the gauge theory. We expect the gauge-matrix duality to fail for such vacuum, but we are not quite sure, and we certainly do not know the specifics (if any) of this failure.

Finally, we would like to build a matrix model of a chiral $(\mathbf{n}, \overline{\mathbf{m}})$ bifundamental with $n \neq m$. As discussed in the introduction, this calls for complex $\hat{N} \times \hat{M}$ matrices integrated over some variety $\Gamma \subset \mathbb{C}^{\hat{N}\hat{M}}$ which has real dimension $\hat{N}\hat{M}$ and satisfies the symmetry condition (1.1). More generally, for a theory with several chiral bifundamentals, one may use correlated matrices: instead of independent matrices $\mathcal{M}_1, \ldots, \mathcal{M}_K$ each integrated over a separate variety Γ_ℓ ($\ell = 1, \ldots, K$), the array $(\mathcal{M}_1, \ldots, \mathcal{M}_K)$ of all the matrices is integrated over a combined variety $\Gamma \subset \mathbb{C}^D$ where $D = \sum_\ell \hat{N}_\ell \hat{M}_\ell$. Such combined variety should have real dimension D (same as the net dimension of all the Γ_ℓ) and satisfy the generalized symmetry condition

$$\forall g \in (\text{net symmetry group}), \qquad g : \Gamma \mapsto \Gamma' \cong \Gamma\,. \tag{5.3}$$

We would like to construct such a variety for an interesting quiver theory, and then compare the loop equations of the matrix model to the anomaly equations for the field theory's chiral ring.

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